Self-focusing and defocusing of twisted light in non-linear media

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Abstract: We study the self-focusing and defocusing of a light beam carrying angular momentum (called twisted light) propagating in a nonlinear medium. We derive a differential equation for the beam width parameter f as a function of the propagation distance, angular frequency, beam waist and intensity of the beam. The method is based on the Wentzel-Kramers-Brillouin and the paraxial approximations. Analytical expressions for f are obtained, analyzed and illustrated for typical experimental situations.

OCIS codes: (260.5950) Self-focusing; (190.4350) Nonlinear optics at surfaces.

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Received 24 Sep 2010; accepted 31 Oct 2010; published 16 Dec 2010

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1. Introduction

The self-focusing of light beams is a basic phenomena in nonlinear optics [1] with a variety of important applications [2] that rely on the manipulation and control of the photon beam [3]. Generally, the theory of self focusing is well established with the propagation characteristics found to be closely related to the properties of the medium [1–4] and to the pulse width of laser beams [5]. Self-focusing and de-focusing of electromagnetic beams in nonlinear media was reviewed by Akhmanov et. al [6]. Recently, several investigations were conducted to study the propagation properties of Cosh Gaussian and Hermite Gaussian beams in different media [7, 8]. Here, we investigate the self focusing of Laguerre Gaussian beams [9–12] in a nonlinear medium using the the Wentzel-Kramers-Brillouin (WKB) and the paraxial approximations [6, 13]. Laguerre Gaussian (LG) beams with a central hole singularity have been shown to play an important role in several areas of optics [14, 15]. In particular, light carrying orbital angular momentum l, also called twisted light, is described by LG modes with a term describing the on-axis phase singularity of strength l; hence the name optical vortices [16–19] for this type of intensity distribution. In addition to the winding number l, LG modes are characterized by their radial index p and their waist size w_0. Here we use LG modes with p = 0; for $l \neq 0$. In this case, the intensity cross-sections perpendicular to the propagation direction consists of one bright ring with no on-axis intensity. This feature makes them ideal for applications in optical trapping and optical tweezers. Furthermore, as LG beams can transfer orbital momentum to the trapped particle, it can also act with a torque on the trapped particle [16–19]. LG tweezers can also trap metallic particles with a refractive index higher than that of the surrounding medium [20, 21].

All of these applications rely on the light scattering and hence they are related to the strength and the distribution of the intensity. The focusing and defocusing are thus important, e.g. in the above context they allow the manipulation of the trapping spot size and the strength of the tweezers.

The paper is structured as follows: starting from the amplitude distribution of LG beams propagating in a nonlinear dielectric medium in section 2, we derive a general differential equation for the beam width parameter. Utilizing the WKB and paraxial approximations in section 3 we derive analytical expression for the intensity distribution as a function of the beam's parameters. In section 4 results are presented graphically and discussed. A brief conclusion and future prospectives are given in section 5.

#135619 - $15.00 USD Received 24 Sep 2010; accepted 31 Oct 2010; published 16 Dec 2010

2. Theoretical background

2.1. Laguerre Gaussian beams

The amplitude distribution of the LG beam $u_{lp}^{LG}(r,\phi,z)$ in a cylindrical coordinate with z axis being along the beam propagation direction, is [24]

$$u_{lp}^{LG}(r,\phi,z) = \frac{C_p^l}{w(z)}\left[\frac{\sqrt{2}r}{w(z)}\right]^l \exp\left[\frac{-r^2}{w^2(z)}\right] L_p^l(\frac{2r^2}{w^2(z)}) \times \exp\left[-i\frac{kr^2z}{2(z^2+z_R^2)} - il\phi + i(2p+l+1)\arctan(\frac{z}{z_R})\right], \quad (1)$$

where r is the radial coordinate and ϕ is the azimuthal angle. $w(z) = w_0\sqrt{1+(z^2/z_R^2)}$ is the radius of the beam at z, and z_R is the Rayleigh range. w_0 is the beam waist at $z=0$. $L_p^l(x)$ is the associated Laguerre Polynomial, C_p^l is the normalization constant, and $(2p+l+1)\arctan(\frac{z}{z_R})$ is the Gouy phase. At the beam waist, $z=0$, the amplitude of a Laguerre- Gaussian beam simplifies to

$$u_{lp}^{LG}(r,\phi,z=0) = C_p^l\left[\frac{\sqrt{2}r}{w_0}\right]^l \exp\left[\frac{-r^2}{w_0^2}\right] L_p^l(\frac{2r^2}{w_0^2})\exp(-il\phi). \quad (2)$$

2.2. Self-focusing and defocusing in a nonlinear medium

We consider a nonlinear medium characterized by the dielectric function $\varepsilon = \varepsilon_0 + F(EE^*)$, i.e. $\varepsilon(r,z)$ depends upon the beam irradiance; the functional dependence of F is determined by the physical situation/mechanism under study. In turn $|E|^2$ depends on z in a manner yet to be determined. In the spirit of the paraxial approximation, we expand F in a Taylor series in powers of r^2 and retain terms up to r^2. This leads to

$$\varepsilon(r,z) \approx \varepsilon_0(z) - r^2\varepsilon_2(z). \quad (3)$$

In the wave equation governing the propagation of the laser beam

$$\nabla^2 E + \frac{\omega^2}{c^2}\varepsilon E + \nabla\left(\frac{E\,\nabla\varepsilon}{\varepsilon}\right) = 0 \quad (4)$$

the third term can be neglected if $k^{-2}\nabla^2(\ln\varepsilon) \ll 1$, where k is the wave vector. This inequality is satisfied in almost all cases of practical interest. For a cylindrically symmetric beam a solution for

$$\nabla^2 E + \frac{\omega^2}{c^2}\varepsilon E = 0 \quad (5)$$

we obtain using WKB and the paraxial approximation Refs. [6, 13] as

$$E(r,\phi,z) = A(r,\phi,z)\exp[i(\omega t - kz)] \quad (6)$$

where $k = \frac{\omega}{c}\sqrt{\varepsilon_0}$ and $A(r,\phi,z)$ is the complex amplitude of the electric field. Substituting for $E(r,\phi,z)$ and neglecting $\frac{\partial^2 A}{\partial z^2}$ on the basis of WKB approximation which implies that the characteristic distance of intensity variation is much greater than wavelength. We obtain

$$2ik\frac{\partial A}{\partial z} = \frac{\partial^2 A}{\partial r^2} + \frac{1}{r}\frac{\partial A}{\partial r} + \frac{1}{r^2}\frac{\partial^2 A}{\partial \phi^2} - \frac{\omega^2}{c^2}\varepsilon_2 r^2 A. \quad (7)$$

To solve Eq. (7) we express $A(r,\phi,z)$ as

$$A(r,\phi,z) = A_0(r,z)\exp\{i[-kS(r,z) - l\phi]\} \tag{8}$$

where A_0 and S are real functions of r, ϕ and z and the Eikonal S is

$$S = \frac{r^2}{2}\beta(z) + \Theta(z). \tag{9}$$

$\Theta(z)$ is an additive function where

$$\beta(z) = \frac{1}{f}\frac{df}{dz}. \tag{10}$$

The parameter $\beta(z)$ is the curvature of the wavefront. Substituting for $A(r,\phi,z)$ and S from Eq. (9) and Eq. (10) in Eq. (8) one obtains

$$2\frac{\partial S}{\partial z} + \left(\frac{\partial S}{\partial r}\right)^2 = \frac{1}{k^2 A_0}\left[\frac{\partial^2 A_0}{\partial r^2} + \frac{1}{r}\frac{\partial A_0}{\partial r} - \frac{l^2}{r^2}A_0\right] - \frac{\varepsilon_2}{\varepsilon_0}r^2 \tag{11}$$

$$\frac{\partial A_0^2}{\partial z} + \frac{\partial A_0^2}{\partial r}\frac{\partial S}{\partial r} + A_0^2\left(\frac{\partial^2 S}{\partial r^2} + \frac{1}{r}\frac{\partial S}{\partial r}\right) = 0 \tag{12}$$

The solution of Eq. (11) for LG beam can be written as

$$A_0(r,z) = \frac{E_0}{f}\left(\frac{\sqrt{2}r}{w_0 f}\right)^l \exp\left(\frac{-r^2}{w_0^2 f^2}\right) L_P^l\left(\frac{2r^2}{w_0^2}\right). \tag{13}$$

For $l = 1$ and $p = 0$, substituting for S and A_0 from Eqs. (9), (13), (11) yields

$$\frac{1}{f}\frac{d^2 f}{dz^2} = \frac{4c^2}{\omega^2 \varepsilon_0(z) w_0^4 f^4} - \frac{\varepsilon_2(z)}{\varepsilon_0(z)}. \tag{14}$$

Equation (14) can always be solved by considering the conditions

$$f = 1 \quad \text{and} \quad \frac{df}{dz} = 0 \text{ at } z = 0. \tag{15}$$

It is, however, convenient to reduce Eq. (14) to a dimensionless form by transforming the coordinate z to the dimensionless distance of propagation

$$\xi = \frac{zc}{w_0^2 \omega} \tag{16}$$

and the beam width w_0 to the dimensionless beam width

$$\rho = \frac{w_0 \omega}{c} \tag{17}$$

Substituting Eq. (16) and (17) in Eq. (14) yields

$$\frac{\varepsilon_0(z)}{f}\frac{d^2 f}{d\xi^2} = \frac{4}{f^4} - \rho^2 w_0^2 \varepsilon_2(z). \tag{18}$$

In case of a parabolic nonlinearity, that is when the nonlinear term is proportional to E^2 we have the r dependent term

$$\varepsilon_2(f) r^2 = \frac{\alpha E_0^2}{f^2}\frac{r^2}{w_0^2 f^2} \tag{19}$$

where α is a constant. Substitution of $\varepsilon_2(f)$ in Eq. (18) yields

$$\frac{1}{f}\frac{d^2 f}{d\xi^2} = \frac{1}{\varepsilon_0 f^4}(4-\rho^2\alpha E_0^2). \tag{20}$$

The analytical solution of Eq. (20) under the conditions (15) is

$$f = \sqrt{\frac{\varepsilon_0 + 4\xi^2 - E_0^2\alpha\xi^2\rho^2}{\varepsilon_0}}. \tag{21}$$

For further analysis it is useful to write Eq. (20) in the form

$$\frac{d^2 f}{dz^2} = \frac{c^2}{\varepsilon_0 w_0^4\omega^2}\left(4 - \frac{\alpha w_0^2\omega^2}{c^2}E_0^2\right) f^{-3}. \tag{22}$$

3. Results and discussions

Equation (20), or (22) is the fundamental second order differential equation governing self-focusing/defocusing of LG beams in a parabolic medium. Essentially, the effect of the non linearity is dictated by the second term on the right of Eq. (20), or (22). In the absence of this term $\frac{d^2 f}{d\xi^2}$ remains positive causing the beam width parameter (f) to increase continuously leading to a steady divergence. This effect is the natural diffraction divergence. The second term containing the nonlinear effect is negative and acts in the opposite direction tending to converge the beam. The convergence (focusing) or divergence (defocusing) of the beam depends on which of the two terms predominates. Equation (22) makes also clear that for the focusing the product $(w_0\omega E_0)^2$ is relevant, i.e. for a focused beam the frequency has to be increased when the intensity (or the waist) is lowered to maintain focusing.

Figures 1(a) and 1(b) illustrate the focusing effects for typical experimentally feasible parameters.

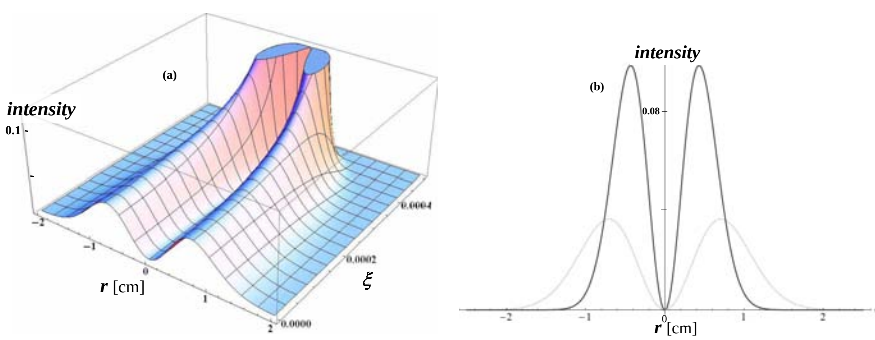

Fig. 1. (color online) (a) Intensity in CGS units of LG beam verses the radial distance from the propagation direction (in cm) for the $l = 1$, and $p = 0$. The angular frequency is $\omega = 2\times10^{14}$ rad/sec, $w_0 = 1\,[cm]$, $\varepsilon_0 = 1$, $\alpha = 1$, $E_0 = 0.3\,[StatV/cm]$. $\rho = 0.66\times10^4$. (b) Initial intensity profile (dotted curve) compared to the propagated intensity at $\xi = 4\times10^{-4}$.

As clear from this figure, focusing and intensity increase occur until at certain value of the normalized distance of propagation $(\xi) = 0.00062$, after that de-focusing sets in. This is due to the fact that at higher intensity nonlinear refractive term dominates over the diffractive term for some initial distance of propagation after that the diffractive term strongly overcomes the nonlinear refractive term and therefore the beam defocuses.

#135619 - $15.00 USD Received 24 Sep 2010; accepted 31 Oct 2010; published 16 Dec 2010

From Fig. 1 it is also obvious that the focusing effect can be utilized, e.g. for creating tighter and stronger three-dimensional optical traps by crossing two LG beams at the focused distance. The predicted focusing effect can also be used for the realization of more versatile optical tweezers. From Eq. (22) we infer that to achieve results similar to those in Fig. 1 for a smaller staring waist one has either to increase the intensity (or the frequency) by the roughly the same amount.

4. Conclusions

We studied the self focusing of a twisted light beam in a nonlinear dielectric medium by using the paraxial approximation. The differential equation for the beam waist is solved analytically. The occurrence of the focussing is pointed out and its dependence on the beam's parameters is worked out analytically and illustrated by numerical calculations. Some practical applications of the predicted effect are pointed out.

#135619 - $15.00 USD
Received 24 Sep 2010; accepted 31 Oct 2010; published 16 Dec 2010