

Equilibrium spin currents and the magnetoelectric effect in magnetic nanostructures

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(Received 26 August 2005; published 7 December 2005)

We discuss the problem of equilibrium spin currents in ferromagnets with inhomogeneous magnetization. Using simple microscopic models we explain the physical origin of equilibrium spin currents. Next we derive the equilibrium spin current from the Hamiltonian with a gauge field associated with local rotations in the spin space. Several examples of magnetic systems are studied in detail, and the persistent spin current is found to exist in the ground state of these systems. We also demonstrate the possibility of measuring the equilibrium spin current using the magnetoelectrically induced electric field near the ring.

DOI: 10.1103/PhysRevB.72.241302

PACS number(s): 72.25.-b, 73.23.Ra, 75.80.+q, 75.75.+a

The problem of generation of the pure spin currents has attracted much attention recently. It is mostly related to the perspectives of possible applications in spintronics, for which the generation and manipulation of spin currents is of prime importance.¹ Several different ways have been proposed to create the spin currents, such as, for example, the injection of spin-polarized carriers from a magnetic metal or semiconductor,² optical excitation of spin-polarized electrons or holes,³ equilibrium spin currents induced by the spiral states in ferromagnets,⁴ spin Hall effect,⁵ spin transport in presence of the spin-orbit (SO) interaction,⁶ and others.

Recently, the transport of magnetization by magnons has been studied by Meier *et al.*⁷ They demonstrated that by using a finite-length spin chain between magnetic reservoirs, the pure spin current can be generated without the transport of electron charge. Then Schütz *et al.* proposed to use the magnons in a mesoscopic Heisenberg ring under an inhomogeneous magnetic field as a possible way to induce the persistent spin current.⁸ It was also found in these works^{7,8} that the magnetization transport by magnons is accompanied by an electric polarization, which can be experimentally observed in the vicinity of the magnetic wire⁷ or the mesoscopic ring.⁸ Most recently, Katsura *et al.* used an electronic model of two transition atoms mediated by an oxygen atom to show that the equilibrium spin current in this system is related to the electric polarization through a mechanism of the magnetoelectric effect,⁹ which, in fact, is much stronger than the previously discussed^{7,8} relativistic mechanism.

At the same time, the theoretical studies of the magnetization transport provoked a lot of discussions about the existence and physical meaning of the nonvanishing equilibrium spin currents and about a proper definition of the spin current in the nonequilibrium transport phenomena.¹⁰⁻¹² The main idea of these works was to redefine the transport spin current, which does not include the equilibrium (supercurrent) part responsible for the spin torque.¹⁰

Here we show that several key points concerning the equilibrium spin currents and electric polarization discussed in the above-mentioned papers^{7-9,12} should be further clarified. In particular, we emphasize that the equilibrium spin current (also called supercurrent because it is associated with the rigidity of the order parameter¹³) is related to a noncol-

linear magnetic ordering, and it exists at $T=0$ without magnons (in contrast to the claims of Refs. 8 and 12). The details of electronic structure and/or SO interaction are not important for the equilibrium spin currents. On the other hand, the observable electric polarization is induced by the equilibrium spin current as well as by the magnon transport, and the corresponding mechanism necessarily includes the spin-orbit interaction. The magnitude of this effect strongly depends on the electronic structure of the material, so that it can be much larger than a bare relativistic effect.⁹ We believe that the effective enhancement of the SO coupling for the magnetoelectric mechanism in the noncollinear magnets is an analog of the renormalization (giant enhancement) of the Rashba SO interaction in the layered systems, as well as the reinforcement of the anomalous Hall effect, which is also related to the SO interaction.

Two spins in local fields. We start from a simple example demonstrating that the equilibrium spin current does not vanish for two interacting quantum spins $S=1/2$. Namely, we consider the Hamiltonian with interacting spins S_1 and S_2 in local magnetic fields \mathbf{B}_1 and \mathbf{B}_2

$$H = -JS_1 \cdot S_2 - \mathbf{B}_1 \cdot S_1 - \mathbf{B}_2 \cdot S_2. \quad (1)$$

The equation of motion of the spin operator $\dot{S}_{1\mu} = (i/\hbar)[H, S_{1\mu}]$ with (1), yields $\dot{S}_1 = (J/\hbar)S_1 \times S_2 + (1/\hbar)S_1 \times \mathbf{B}_1$. This equation has the meaning of a hydrodynamic equation for spin flow, with the first term on the right-hand side being the spin current flowing from 2 to 1,

$$\mathbf{j}_{2 \rightarrow 1} \equiv \frac{J}{\hbar} S_1 \times S_2 = -\mathbf{j}_{1 \rightarrow 2}, \quad (2)$$

and the second being a local source (or sink) term. In the equilibrium state, $\langle \dot{S}_1 \rangle = \langle \dot{S}_2 \rangle = 0$ due to compensation of the moments transferred by the spin current [due to the mutual noncollinearity of S_1 and S_2 (Ref. 14)] and generated by the local source terms due to the noncollinearity of S_1 and S_2 with the local fields \mathbf{B}_1 and \mathbf{B}_2 , respectively.

Hubbard model. This result can be explained by microscopic calculation using a two-site electronic model with

Coulomb interaction. Let us consider a model of a two-electron system with strong on-site Hubbard interaction U . This is an example of an electronic system, which is often used as a model leading to the localized magnetic moments with antiferromagnetic interaction. We show that by applying on-site magnetic fields \mathbf{B}_1 and \mathbf{B}_2 with different orientations, we induce the spin current transferring a spin moment. The Hamiltonian is

$$H = -t(c_{1\alpha}^\dagger c_{2\alpha} + c_{2\alpha}^\dagger c_{1\alpha}) + \sum_{i=1,2} (Un_i^\dagger n_i - \mathbf{B}_i \cdot \mathbf{S}_i), \quad (3)$$

where t is the hopping, $c_{i\alpha}^\dagger$ and $c_{i\alpha}$ are the creation and annihilation operators for electrons with spin α at the site i , $n_i^\alpha \equiv c_{i\alpha}^\dagger c_{i\alpha}$ is the occupation number operator for site i and spin α , and $\mathbf{S}_i \equiv \frac{1}{2} c_{i\alpha}^\dagger \boldsymbol{\sigma}_{\alpha\beta} c_{i\beta}$ is the spin operator at site i ; here and throughout the paper, we use the summation convention for repeated indices. We consider the case of strong on-site interaction and weak field, $B \ll t \ll U$.

The spin dynamics is determined by $\dot{\mathbf{S}}_1 = (i/\hbar)[H, \mathbf{S}_1]$. Using (3) we find

$$\dot{\mathbf{S}}_1 = \frac{it}{2\hbar} (c_{1\alpha}^\dagger c_{2\beta} - c_{2\alpha}^\dagger c_{1\beta}) \boldsymbol{\sigma}_{\alpha\beta} + \frac{1}{\hbar} \mathbf{S}_1 \times \mathbf{B}_1. \quad (4)$$

The first term, on the right-hand side, is the spin current operator

$$\mathbf{j}_{2 \rightarrow 1} \equiv \frac{it}{2\hbar} (c_{1\alpha}^\dagger c_{2\beta} - c_{2\alpha}^\dagger c_{1\beta}) \boldsymbol{\sigma}_{\alpha\beta} = -\mathbf{j}_{1 \rightarrow 2}. \quad (5)$$

We shall now explain how this result is related to the one obtained above for the two-spin Heisenberg model. Using the usual procedure of unitary transformation, we approximately (to order t^3/U^2 or Bt^2/U^2) decouple the four (singly occupied) low energy states from the two (doubly occupied) high energy states. The effective Hamiltonian for the four low energy states is $\tilde{H} = -J(\mathbf{S}_1 \cdot \mathbf{S}_2 - 1/4) - \mathbf{B}_1 \cdot \mathbf{S}_1 - \mathbf{B}_2 \cdot \mathbf{S}_2$, where $-J \equiv 4t^2/U$. By applying the same unitary transformation to the spin current operator, we obtain

$$\tilde{\mathbf{j}}_{2 \rightarrow 1} \equiv \frac{J}{\hbar} \mathbf{S}_1 \times \mathbf{S}_2, \quad (6)$$

which coincides with the result obtained for the two-spin Heisenberg model. This result explicitly demonstrates that the underlying physical mechanism for the spin current between noncollinear spins is the same as for the exchange coupling, namely the back and forth hopping of electrons between neighboring spins.

Noncollinear ferromagnet. Now we consider a textured ferromagnet, which represents a magnetic system in a topologically nontrivial metastable state. A simple example of such a system is a magnetic ring with the easy-axis magnetic anisotropy, so that the magnetic moment, oriented along the ring, creates a vortex.

Let us take a continuous model of a ferromagnet described by the Hamiltonian, which includes the exchange interaction, anisotropy, and the interaction with an external magnetic field $\mathbf{B}(\mathbf{r})$ (this model was used recently to determine the Berry phase of magnons in textured ferromagnets¹⁵)

$$H = \int d^3\mathbf{r} \left[\frac{a}{2} (\partial_i n_\mu)^2 + \mathcal{F}\{\mathbf{n}(\mathbf{r})\} - \beta B_\mu n_\mu \right], \quad (7)$$

where $\mathbf{n}(\mathbf{r})$ is the unit vector oriented along the magnetization at the point \mathbf{r} , a is the constant of exchange interaction, $\mathcal{F}\{\mathbf{n}(\mathbf{r})\}$ is a function determining the anisotropy.

We will use the definition of spin current related to the transformation of the Hamiltonian under local rotations of vector \mathbf{n} in the spin space.¹³ This definition is in accordance with a general definition of currents in the quantum field theory.¹⁶ In the case of a classical magnetic system with Hamiltonian (8), the corresponding transformations of the vector $\mathbf{n}(\mathbf{r})$ belong to the group SO(3). Thus, we perform a local rotation $\mathbf{n}(\mathbf{r}) \rightarrow \mathbf{R}(\mathbf{r})\mathbf{n}(\mathbf{r})$ using the orthogonal transformation matrix $\mathbf{R}(\mathbf{r}) = e^{i\psi(\mathbf{r})J^z} e^{i\theta(\mathbf{r})J^y} e^{i\phi(\mathbf{r})J^x}$, where ψ , θ , ϕ are the Euler angles determining the arbitrary rotations of the coordinate frame, and J^x , J^y , and J^z are the generators of three-dimensional rotations around the x , y , and z axes, respectively.

The Hamiltonian of exchange interaction [the first term in Eq. (8)] in the rotated frame has the form

$$H_{ex} = \frac{a}{2} \int d^3\mathbf{r} (\partial_i n_\mu - A_{i\mu\nu} n_\nu)^2, \quad (8)$$

where the gauge field $A_i(\mathbf{r}) = (\partial_i \mathbf{R})\mathbf{R}^{-1}$. The matrix $A_i(\mathbf{r})$ can be presented as $A_i(\mathbf{r}) = iJ^\mu \mathcal{A}_i^\mu(\mathbf{r})$, where $\mathcal{A}_i^\mu(\mathbf{r})$ belongs to the adjoint representation of the group SO(3). Then the exchange energy can be written as

$$H_{ex} = \frac{a}{2} \int d^3\mathbf{r} [(\partial_i \delta_{\alpha\beta} - iA_i^\mu J_{\alpha\beta}^\mu) n_\beta]^2, \quad (9)$$

and the spin current density is defined as $j_i^\mu = \gamma(\delta H / \delta \mathcal{A}_i^\mu)$, where γ is the gyromagnetic ratio. We find

$$j_i^\mu = -ic_s J_{\alpha\beta}^\mu n_\beta (\partial_i \delta_{\alpha\gamma} - iA_i^\delta J_{\alpha\gamma}^\delta) n_\gamma, \quad (10)$$

where $c_s = \gamma a$. The spin current (11) is gauge invariant. We can fix the gauge by taking the auxiliary field $\mathcal{A}_i^\mu = 0$. Then, using the relation $iJ_{\nu\lambda}^\mu = \epsilon_{\mu\nu\lambda}$, we finally obtain

$$\mathbf{j}_i = c_s \mathbf{n} \times \partial_i \mathbf{n}. \quad (11)$$

Thus, the spin current¹² is nonzero in the noncollinear ferromagnets. In particular, it is nonzero in a metastable state of the ferromagnet with topological excitations.¹⁷ This definition is consistent with the spin-conservation equation relating the variation of the magnetization in time to the divergence of the spin current,^{8,18} $\dot{n}_\mu + \text{div } \mathbf{j}^\mu = T_\mu$, where $T_\mu = \gamma \epsilon_{\mu\nu\lambda} n_\nu B_\lambda^{\text{eff}}$, and \mathbf{B}^{eff} is the local effective magnetic field, comprising the local external field and other local terms, such as magnetic anisotropies. The physical meaning is that of a hydrodynamic equation for $\mathbf{n}(\mathbf{r})$, where T_μ is a local source term (torque).

The thermodynamic average of the spin current is

$$\langle j_i^\mu \rangle = c_s \epsilon_{\mu\nu\lambda} [\langle n_\nu \rangle \partial_i \langle n_\lambda \rangle + \langle \delta n_\nu \partial_i \delta n_\lambda \rangle], \quad (12)$$

where $\langle X \rangle \equiv \text{Tr}(Xe^{-\beta H}) / \text{Tr}(e^{-\beta H})$, and $\delta X = X - \langle X \rangle$ is the fluctuation of X . Thus, there are two contributions to the spin current: the principal one arises from spatial variations of the

average value of the magnetization axis, while the second one is due to the magnetization fluctuations (magnons). The magnon term is likely to be a small correction as compared to the principal (ground-state) term. The ground-state term can be interpreted as the supercurrent associated with the rigidity of the order parameter.^{4,13,19} The magnon term has been discussed thoroughly,^{7,8,20} and will not be discussed further here. In the following, we assume $T=0$, so that the magnon term vanishes (except for the effect of quantum zero-point fluctuations, which are usually negligibly small in ferromagnets). In this case, we can drop the angular brackets, which means that the thermodynamic average is implied.

Mesoscopic ring. To calculate the spin current in the ring geometry, it is convenient to use the cylindrical coordinates (ρ, φ, z) for the point on the ring. We assume that the vector \mathbf{n} does not depend on ρ and z . Then the exchange Hamiltonian H_{ex} can be written as

$$H_{ex} = \frac{a\zeta_0}{2R} \int_0^{2\pi} d\varphi \left(\partial_\varphi n_\mu + \frac{1}{2\pi} \epsilon_{\mu\lambda\nu} \Phi_\lambda n_\nu \right)^2, \quad (13)$$

where R and ζ_0 are the radius and the cross section of the ring, respectively, $\Phi_\lambda \equiv L A_\varphi^\lambda$, and $L=2\pi R$.

Let us take A_φ^μ constant along the ring. Then Φ_μ is the flux of the μ component of the gauge field A_φ^μ through the ring. Now we get the azimuthal spin current density in the ring as $j_\varphi^\mu = (\gamma/\zeta_0)(\partial H/\partial \Phi_\mu)$. Using (14) and taking $\Phi_\mu=0$, we find

$$j_\varphi^\rho = \frac{c_s}{L} \int d\varphi [-n_z(n_\rho + \partial_\varphi n_\rho) + n_\varphi \partial_\varphi n_z], \quad (14)$$

$$j_\varphi^\varphi = \frac{c_s}{L} \int d\varphi [n_z(-n_\varphi + \partial_\varphi n_\rho) - n_\rho \partial_\varphi n_z], \quad (15)$$

$$j_\varphi^z = \frac{c_s}{L} \int d\varphi [n_\varphi(n_\varphi - \partial_\varphi n_\rho) + n_\rho(n_\rho + \partial_\varphi n_\varphi)]. \quad (16)$$

In particular cases of the tangential magnetization ($n_\varphi=1, n_\rho=n_z=0$), radial magnetization ($n_\rho=1, n_\varphi=n_z=0$), and also for any intermediate case with the in-plane magnetization making a constant angle with the tangent vector, using (15)–(17) we obtain

$$j_\varphi^z = 2\pi c_s/L, \quad j_\varphi^\rho = j_\varphi^\varphi = 0. \quad (17)$$

It should be emphasized that the spin current (18) is related to the assumed metastable state of magnetization field but not to the magnons. The contribution of magnons exists for $T \neq 0$ but it is small because the magnons are weak excitations over the metastable state, completely vanishing in the limit of $T \rightarrow 0$.

The other example is a magnetic ring with uniaxial anisotropy in a homogeneous magnetic field along the axis z . Due to the anisotropy and exchange interaction, the magnetization along the ring is oriented with a certain angle out of the ring plane. There exist a metastable state with crownlike magnetization profile in this system. (Due to the dipolar forces it can be the ground state.) We can calculate the angle θ using Hamiltonian (8) with $\mathcal{F}\{\mathbf{n}(\mathbf{r})\} = \lambda n_z^2/2$ and $\lambda > 0$, and

the magnetic field \mathbf{B} oriented along the axis z . Then, using the polar coordinates and assuming that n_φ and n_z do not depend on the coordinate along the ring, we find the energy of the metastable state

$$E = \pi\zeta_0 R [a n_\varphi^2/(2R^2) + \lambda n_z^2/2 - \beta B n_z] \quad (18)$$

and calculate the angle θ minimizing the energy (19),¹⁵ $\cos \theta = \max\{1, \beta B/(\lambda - a/R^2)\}$ for $\lambda > a/R^2$, and $\theta=0$ for $\lambda \leq a/R^2$. The spin current in the ground state with $\theta \neq 0$ does not vanish. We calculate the components of it using Eqs. (15)–(17)

$$j_\varphi^\rho = \frac{2\pi c_s}{L} \sin \theta \cos \theta, \quad j_\varphi^z = \frac{2\pi c_s}{L} \sin^2 \theta, \quad j_\varphi^\varphi = 0. \quad (19)$$

Obviously, we obtain the same value of the spin current for different magnetization profiles if they can be transformed to each other by a global rotation.

Electric polarization. As discussed in Refs. 7 and 8, the spin current implies an electric polarization $P_i = \epsilon_{ij\mu} j_j^\mu / c^2$, which is the relativistic effect of a transformation of magnetization to the electric field in the moving frame (this fact has been known long ago²¹). The important point is that the electric polarization appears not only due to the spin transport of magnons but also due the spin supercurrent in the ground state [first term in Eq. (13)]. This was shown by Katsura *et al.*⁹ using a microscopic model with two transition atoms and an oxygen atom in between. Their model calculations demonstrate that the magnetoelectric coefficient determining the magnitude of effect depends essentially on the material parameters. Taking it into account, we can write the polarization

$$\mathbf{P} = \alpha_{me} [\mathbf{n}(\nabla \cdot \mathbf{n}) - (\mathbf{n} \cdot \nabla)\mathbf{n}], \quad (20)$$

where we denoted by α_{me} the magnetoelectric coefficient. This equation implies an enhancement of the bare relativistic effect in the condensed matter, i.e., it is due to the proposed magnetoelectric effect.⁹ It should be noted that the suitable materials for the observation of polarization are magnetic insulators and semiconductors because the induced electric polarization in a good metal would be completely screened by free electrons.

In the cases of the magnetic ring with the tangent magnetization along the ring or the constant radial magnetization, using (21) we find that the polarization vector \mathbf{P} is oriented along the radius of the ring, and $P_r = -\alpha_{me}/R$, where R is the ring radius.

Let us estimate the magnitude of effect for a typical magnetic nanostructure, namely, a ferromagnetic disc of radius R and thickness h with a circular vortex domain (tangential magnetization).²² The effect of the vortex can be approximated by replacing the disc by a ring of internal radius r equal to the radius of the vortex core. In this case the polarization \mathbf{P} is radial and decays as the inverse distance from the ring axis. We can approximately estimate the electrostatic potential U at a point located on the axis at an altitude z

above the disc in the limit of $r, h \ll |z| \ll R$. We obtain $U \approx -\alpha_{me} h / |z|$. We estimate α_{me} using the result of a three-atom model from Ref. 9 for the polarization per unit volume, which gives $\alpha_{me} \approx (e/a)(V/\Delta)^3$. Here a is the lattice constant, V is a hopping energy (hopping between the transition and oxygen atoms in Ref. 9), and Δ is a characteristic electron energy (between d and p orbitals of different atoms). This formula was obtained for large SO splitting $\Delta_{SO} \gg \Delta$, and it does not explicitly depend the SO interaction. For a magnetic disc, using typical values $R=1 \mu\text{m}$, $h=10 \text{ nm}$, and $r=10 \text{ nm}$, this yields $U \approx 0.1(V/\Delta)^3 \text{ V}$ at an altitude $|z|=100 \text{ nm}$. The vortex can be removed by application of an external magnetic field. The effect can be possibly detected

experimentally by using a single-electron transistor as a detector of electric field.

We calculated the equilibrium spin current in mesoscopic systems, and showed that it does not vanish in the metastable state of magnetic system with topologically nontrivial magnetization profile. The electric polarization induced by the spin current can be experimentally measured.

This work is supported by FCT Grant No. POCI/FIS/58746/2004 (Portugal), and by Polish Grants No. PBZ/KBN/044/P03/2001 and 2 P03B 053 25. V.D. thanks the Calouste Gulbenkian Foundation in Portugal for support. P.B. thanks P. Kopietz for stimulating correspondence.

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