Berry phase of magnons in textured ferromagnets

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We study the energy spectrum of magnons in a ferromagnet with topologically nontrivial magnetization profile. In the case of inhomogeneous magnetization corresponding to a metastable state of ferromagnet, the spin-wave equation of motion acquires a gauge potential leading to a Berry phase for the magnons propagating along a closed contour. The effect of magnetic anisotropy is crucial for the Berry phase: we show that the anisotropy suppresses its magnitude, which makes the Berry phase observable in some cases, similar to the Aharonov-Bohm effect for electrons. For example, it can be observed in the interference of spin waves propagating in mesoscopic rings. We discuss the effect of domain walls on the interference in ferromagnetic rings, and propose some experiments with a certain geometry of magnetization. We also show that the non-vanishing average topological field acts on the magnons like a uniform magnetic field on electrons. It leads to the quantization of the magnon spectrum in the topological field.

I. INTRODUCTION

The Berry phase theory[1–3] allowed generalization of the idea of Aharonov-Bohm effect[4] on electrons in the electromagnetic potential, to an analogous effect related to a gauge potential, which arises during the adiabatic motion of a quantum system in a parametric space. Up to now, much effort has been directed to better understand and find an experimental confirmation for the Berry phase of electrons, for example, in the case of electrons moving in a varying magnetization field of the inhomogeneous ferromagnet.[5,6]

One of the most intriguing consequences of the Berry phase theory is the possibility of the Aharonov-Bohm-type effect on electrically neutral particles or boson fields.[7,8] An example of the adiabatic phase for the polarized light has been investigated by Pancharatnam[9] and Berry.[10] The other example is the Aharonov-Bohm effect for the exciton,[11] which is a bound state of an electron and a hole in semiconductors.

Here, we consider the effect of the gauge potential and Berry phase on the propagation of magnons in textured ferromagnets. Such quasiparticles are usually viewed as the elementary excitations of the ordered homogeneous state of a ferromagnet, but they can also be used to classify the excited states near a metastable inhomogeneous magnetic configuration. These magnons describe the dynamics of weakly excited inhomogeneous ferromagnet.

The dynamics of magnetization in nanomagnets has been the focus of recent activity[12] because of its importance for magnetoelectronic applications.[13,14] It includes the switching of magnetization by electric current, spin pumping, magnetization reversal in microscopic spin valves, etc. Usually, the magnons play a negative role in the magnetization dynamics limiting the frequency of magnetic reversal, and also leading to the energy dissipation. However, they can probably be used in the spin-transport phenomena such as the spin currents of magnetically polarized electrons.

Here, we study the energy spectrum of spin waves in ferromagnets with a static inhomogeneous magnetization pro-file, and we demonstrate the possibility of observation of the Berry phase in the interference experiments on spin waves in magnetic nanostructures. Recent results of the micromagnetic computer simulation[15–17] of such systems demonstrate that the interference of spin waves actually can be observed in magnetic nanorings with domain walls.

The equation for spin-wave excitations in a general case of arbitrary local frame, depending on both coordinate and time, was found long ago by Korenman *et al.*[18] in the context of local-band theory of itinerant magnetism.[19] Here, we use an idea of this method to relate the adiabatic space transformation to the Berry phase, and to find corresponding properties of the spin waves in a topologically nontrivial inhomogeneous magnetic profile, which is a metastable state of the ferromagnet. We show that the magnetic anisotropy is a crucial element determining the possibility of observation of the Berry phase in real experiments.

II. MODEL AND SPIN WAVE EQUATIONS

We consider the model of a ferromagnet described by the Hamiltonian, which includes the exchange interaction, anisotropy, and the interaction with external magnetic field. It has the following general form:

$$H = \frac{1}{2} \int d^3\mathbf{r} \left[a\left(\frac{\partial n_\mu(\mathbf{r})}{\partial r_i} \right)^2 + \mathcal{F}\{\mathbf{n}(\mathbf{r})\} \right], \qquad (1)$$

where $\mathbf{n}(\mathbf{r})$ is the unit vector oriented along the magnetization $\mathbf{M}(\mathbf{r})$ at the point $\mathbf{r}$, a is the constant of exchange interaction, $\mathcal{F}\{\mathbf{n}(\mathbf{r})\}$ is a function determining the magnetic anisotropy [correspondingly, it includes a certain number of tensors relating the components of vector $\mathbf{n}(\mathbf{r})$] and the dependence on external field, and M_0 is the magnitude of magnetization.

Due to the condition $\mathbf{n}^2(\mathbf{r})=1$, the model is constrained and belongs to the class of nonlinear σ models.[20] The sta-

tionary (saddle point) solutions for the magnetization vector $\mathbf{n}_0(\mathbf{r})$, describing metastable states of the ferromagnet, can be found by minimizing Hamiltonian (1) with the constraint $\mathbf{n}^2(\mathbf{r}) = 1$. It was shown (see, e.g., Refs. 21 and 22) that such metastable states with inhomogeneous magnetization profile are related to the topology of ferromagnetic ordering, and they can include skyrmions, magnetic vortices, and other topological objects.

We are interested in describing the dynamics of small deviations $\delta\mathbf{n}(\mathbf{r})$ from a certain metastable profile $\mathbf{n}_0(\mathbf{r})$ with a nonuniform magnetization, $\mathbf{n}(\mathbf{r}) = \mathbf{n}_0(\mathbf{r}) + \delta\mathbf{n}(\mathbf{r})$, $|\delta\mathbf{n}(\mathbf{r})| \ll 1$. Correspondingly, we assume that the solution of a saddle-point equation describing the state $\mathbf{n}_0(\mathbf{r})$ is already known.

We perform a local transformation

$$\tilde{\mathbf{n}}(\mathbf{r}) = R(\mathbf{r})\mathbf{n}(\mathbf{r}), \tag{2}$$

using the orthogonal transformation matrix $R(\mathbf{r})$. By definition, it determines the rotation of local frame in each point of the space, so that the magnetization in the local frame is oriented along the z axis, $\tilde{\mathbf{n}}_0 = (0, 0, 1)$. Then, we consider small deviations of magnetization $\mathbf{s}(\mathbf{r})$ from $\tilde{\mathbf{n}}_0$. Since $\mathbf{s}(\mathbf{r})$ is small and vectors $\tilde{\mathbf{n}}_0$ are oriented along z, the vectors $\mathbf{s}(\mathbf{r})$ lie in the $x-y$ plane.

The transformation matrix in Eq. (2) is taken in a general form of orthogonal transformation

$$R(\mathbf{r}) = e^{i\psi(\mathbf{r})J_z} e^{i\theta(\mathbf{r})J_y} e^{i\phi(\mathbf{r})J_z}, \tag{3}$$

where ψ, θ, ϕ are the Euler angles determining an arbitrary rotation of the coordinate frame, and J_x, J_y, and J_z are the generators of 3D rotations around the x, y, and z axes, respectively.

Two rotation parameters (for definiteness, the angles θ and ϕ) can be used to define the frame with the z axis along the vector $\mathbf{n}_0(\mathbf{r})$. In the absence of anisotropy, the additional rotation to the angle ψ is purely gauge transformation. However, in a general case of anisotropic system, this rotation allows one to choose the local frame in correspondence with the orientation of anisotropy axes.

The Hamiltonian of exchange interaction [the first term in Eq. (1)] in the rotated frame has the following form:

$$H_{ex} = \frac{a}{2} \int d^3\mathbf{r} \left(\frac{\partial \tilde{n}_\mu}{\partial r_i} - A_i^{\mu\nu} \tilde{n}_\nu \right)^2, \tag{4}$$

where the gauge field $A_i(\mathbf{r})$ is defined by

$$A_i(\mathbf{r}) = \left(\frac{\partial}{\partial r_i} R \right) R^{-1}. \tag{5}$$

Transformation (3) and gauge potential (5) are 3×3 matrices acting on the magnetization vectors. The matrix $A_i(\mathbf{r})$ can also be presented as

$$A_i(\mathbf{r}) = i \mathcal{A}_i^\mu(\mathbf{r}) J_\mu, \tag{6}$$

where $\mathcal{A}_i^\mu(\mathbf{r})$ belongs to the adjoint representation of the rotation group.

Using (3) and (6), we find the explicit dependence of the gauge potential on the Euler angles

$$\mathcal{A}_i^x(\mathbf{r}) = \sin\psi \frac{\partial\theta}{\partial r_i} - \sin\theta \cos\psi \frac{\partial\phi}{\partial r_i},$$

$$\mathcal{A}_i^y(\mathbf{r}) = \cos\psi \frac{\partial\theta}{\partial r_i} + \sin\theta \sin\psi \frac{\partial\phi}{\partial r_i},$$

$$\mathcal{A}_i^z(\mathbf{r}) = \frac{\partial\psi}{\partial r_i} + \cos\theta \frac{\partial\phi}{\partial r_i}. \tag{7}$$

The magnetic anisotropy described by the second term in the right-hand part of (1) gives, after transformation to the local frame, a function $\tilde{\mathcal{F}}\{\tilde{\mathbf{n}}(\mathbf{r})\}$ with correspondingly transformed tensor fields. Here, we do not restrict the general consideration of the problem by any specific form of the anisotropy, but in the following we consider the most important examples of easy-plane- and easy-axis anisotropy.

The Landau-Lifshitz equations for the magnetization in the locally transformed frame are

$$\frac{\partial \tilde{n}_\mu}{\partial t} = -\frac{\gamma}{M_0} \epsilon_{\mu\nu\lambda} \tilde{n}_\nu \left(\frac{\delta H}{\delta \tilde{n}_\lambda} - \nabla_i^{\lambda\rho} \frac{\delta H}{\delta(\nabla_i^{\rho\tau} \tilde{n}_\tau)} \right), \tag{8}$$

where $\epsilon_{\mu\nu\lambda}$ is the unit antisymmetric tensor, and

$$\nabla_i^{\mu\nu} = \frac{\partial}{\partial r_i} \delta_{\mu\nu} - A_i^{\mu\nu} \tag{9}$$

is the covariant derivative. The right-hand part of Eq. (8) vanishes for the magnetization profile corresponding to a metastable state. This is seen from the Landau-Lifshitz equation in the unrotated original frame. In the following, we will use Eq. (8) for the small deviations of magnetization from the metastable state. Hence, we will consider in the right part of (8) only the terms linear in deviations.

Using (1), (4), and (8), we find the equations for weak magnetic excitations near the metastable state (spin waves)

$$\frac{\partial s_x}{\partial t} = -c_s \left[\frac{\partial^2 s_y}{\partial r_i^2} - 2\mathcal{A}_i^z \frac{\partial s_x}{\partial r_i} - (\mathcal{A}_i^z)^2 s_y - \frac{\partial \mathcal{A}_i^z}{\partial r_i} s_x + (\mathcal{A}_i^y)^2 s_y \right.$$
$$\left. + \mathcal{A}_i^x \mathcal{A}_i^y s_x \right] + \frac{\gamma}{M_0} \frac{\partial^2 \tilde{\mathcal{F}}}{\partial \tilde{n}_x \partial \tilde{n}_y} s_x + \frac{\gamma}{M_0} \frac{\partial^2 \tilde{\mathcal{F}}}{\partial \tilde{n}_y^2} s_y, \tag{10}$$

$$\frac{\partial s_y}{\partial t} = c_s \left[\frac{\partial^2 s_x}{\partial r_i^2} + 2\mathcal{A}_i^z \frac{\partial s_y}{\partial r_i} - (\mathcal{A}_i^z)^2 s_x + \frac{\partial \mathcal{A}_i^z}{\partial r_i} s_y + (\mathcal{A}_i^x)^2 s_x \right.$$
$$\left. + \mathcal{A}_i^x \mathcal{A}_i^y s_y \right] - \frac{\gamma}{M_0} \frac{\partial^2 \tilde{\mathcal{F}}}{\partial \tilde{n}_x \partial \tilde{n}_y} s_y - \frac{\gamma}{M_0} \frac{\partial^2 \tilde{\mathcal{F}}}{\partial \tilde{n}_x^2} s_x, \tag{11}$$

where $c_s = \gamma a/M_0$ is the stiffness.

Using (10) and (11), we can also present the equations for circular components of the spin wave, $s_\pm = s_x \pm i s_y$

$$\pm i \frac{\partial s_\pm}{\partial t} = \left[-c_s \left(\frac{\partial}{\partial r_i} \mp i A_i^z \right)^2 - V(\mathbf{r}) + \frac{\gamma}{2M_0} \frac{\partial^2 \widetilde{\mathcal{F}}}{\partial \widetilde{n}_x^2} \right.$$
$$\left. + \frac{\gamma}{2M_0} \frac{\partial^2 \widetilde{\mathcal{F}}}{\partial \widetilde{n}_y^2} \right] s_\pm + \left[-w(\mathbf{r}) - ic_s \mathcal{A}_i^x \mathcal{A}_i^y \right.$$
$$\left. + \frac{i\gamma}{M_0} \frac{\partial^2 \widetilde{\mathcal{F}}}{\partial \widetilde{n}_x \partial \widetilde{n}_y} + \frac{\gamma}{2M_0} \frac{\partial^2 \widetilde{\mathcal{F}}}{\partial \widetilde{n}_x^2} - \frac{\gamma}{2M_0} \frac{\partial^2 \widetilde{\mathcal{F}}}{\partial \widetilde{n}_y^2} \right] s_\mp, \quad (12)$$

where $V(\mathbf{r})$ and $w(\mathbf{r})$ are, respectively, the effective potential and a mixing field acting on the spin wave

$$V(\mathbf{r}) = \frac{c_s}{2} [(\mathcal{A}_i^x)^2 + (\mathcal{A}_i^y)^2], \quad (13)$$

$$w(\mathbf{r}) = \frac{c_s}{2} [(\mathcal{A}_i^x)^2 - (\mathcal{A}_i^y)^2]. \quad (14)$$

Equations (12) for $s_+(\mathbf{r},t)$ and $s_-(\mathbf{r},t)$ are complex conjugates to each other since they both describe the same spin wave with real components $s_x(\mathbf{r},t)$ and $s_y(\mathbf{r},t)$.

We can see that $V(\mathbf{r})$ is an effective potential profile for the propagation of spin wave. Due to the terms $w(\mathbf{r})$ and $ic_s \mathcal{A}_i^x \mathcal{A}_i^y$ in (12), the equations for circular components s_+ and s_- are coupled even in the absence of anisotropy. All these terms are of the second order in derivative of the rotation angle, and they are small in the adiabatic limit, corresponding to a smooth variation of the magnetization vector $\mathbf{n}(\mathbf{r})$.

III. SEMICLASSICAL APPROXIMATION

Equations (10) and (11) can be solved in the semiclassical approximation. The condition of its applicability is a smooth variation of gauge potential $\mathcal{A}_i^\mu(\mathbf{r})$ and fields related to the anisotropy, as well as the external magnetic field, at the wavelength of the spin wave, $kL \gg 1$, where k is the wave vector of the spin wave and L is the characteristic length of the variation of $\mathcal{A}_i^\mu(\mathbf{r})$ and $\mathcal{F}\{\mathbf{n}(\mathbf{r})\}$ (more exactly, the minimum of the corresponding characteristic lengths). Note that the condition of applicability of the semiclassical approximation to solve the spin-wave equations does not require any smallness of the gauge potential itself.

Starting from Eqs. (10) and (11), we look for a general semiclassical solution in the form

$$s_x(\mathbf{r},t) = a \cos[\xi(\mathbf{r}) - \omega t] + b \sin[\xi(\mathbf{r}) - \omega t], \quad (15)$$

$$s_y(\mathbf{r},t) = d \sin[\xi(\mathbf{r}) - \omega t] + f \cos[\xi(\mathbf{r}) - \omega t], \quad (16)$$

with arbitrary coefficients a, b, d, f, and a smooth function $\xi(\mathbf{r})$, so that we can neglect the second derivative of $\xi(\mathbf{r})$ over coordinate $\mathbf{r}$. Substituting (15) and (16) in (10) and (11), we can find four equation for the a, b, d, f coefficients.

The solution (15) and (16) describes the elliptic spin wave with an arbitrary choice of the axes x and y, and, generally, with a varying in space orientation of the principal axes of the ellipse. We can simplify our consideration by choosing

the angle $\psi(\mathbf{r})$ at each point of the space in accordance with the orientation of the principal axes. The corresponding equation for $\psi(\mathbf{r})$ can be found from the condition of $b=f=0$ in Eqs. (15) and (16)

$$c_s \mathcal{A}_i^x \mathcal{A}_i^y - \frac{\gamma}{M_0} \frac{\partial^2 \widetilde{\mathcal{F}}}{\partial \widetilde{n}_x \partial \widetilde{n}_y} = 0. \quad (17)$$

Using (17) and neglecting the terms with derivative of $\mathcal{A}_i^z$, which are small in the semiclassical approximation, we write the spin-wave equations (10) and (11) as

$$\frac{\partial s_x}{\partial t} = -c_s \left[\frac{\partial^2 s_y}{\partial r_i^2} - 2\mathcal{A}_i^z \frac{\partial s_x}{\partial r_i} - (\mathcal{A}_i^z)^2 s_y + (\mathcal{A}_i^y)^2 s_y \right] + \frac{\gamma}{M_0} \frac{\partial^2 \widetilde{\mathcal{F}}}{\partial \widetilde{n}_y^2} s_y, \quad (18)$$

$$\frac{\partial s_y}{\partial t} = c_s \left[\frac{\partial^2 s_x}{\partial r_i^2} + 2\mathcal{A}_i^z \frac{\partial s_y}{\partial r_i} - (\mathcal{A}_i^z)^2 s_x + (\mathcal{A}_i^x)^2 s_x \right] - \frac{\gamma}{M_0} \frac{\partial^2 \widetilde{\mathcal{F}}}{\partial \widetilde{n}_x^2} s_x. \quad (19)$$

Note that by fixing the angle ψ in Eq. (17), we are choosing the gauge, which defines completely the potential $\mathcal{A}_i^\mu$. We do it in spirit of the usual fixing gauge in the Wentzel-Kramers-Brillouin (WKB) approximation.

After substitution of (15) and (16) with $b=f=0$ into (18) and (19), we come to the following equation for the momentum $k_i(\mathbf{r}) \equiv \partial \xi(\mathbf{r}) / \partial r_i$:

$$(\omega + 2c_s \mathcal{A}_i^z k_i)^2 - \{c_s[k_i^2 + (\mathcal{A}_i^z)^2 - (\mathcal{A}_i^y)^2] + 2p_x\}\{c_s[k_i^2 + (\mathcal{A}_i^z)^2$$
$$- (\mathcal{A}_i^x)^2] + 2p_y\} = 0, \quad (20)$$

where

$$p_{x,y}(\mathbf{r}) = \frac{\gamma}{2M_0} \frac{\partial^2 \widetilde{\mathcal{F}}}{\partial \widetilde{n}_{x,y}^2}, \quad (21)$$

are the anisotropy parameters.

Equation (20) should be solved for $k_i(\mathbf{r})$ as a function of smooth inhomogeneous field $\mathcal{A}_i^\mu(\mathbf{r})$. This equation does not constraint the orientation of $\mathbf{k}(\mathbf{r})$ but determines the magnitude of vector $\mathbf{k}(\mathbf{r})$ for each direction in the momentum space. Let us take vector $\mathbf{k}(\mathbf{r})$ along an arbitrary direction, defined by a unity vector $\mathbf{g}$. Then, we can rewrite (20) as

$$[\omega + 2c_s k g_i \mathcal{A}_i^z]^2 - \{c_s[k_i^2 + (\mathcal{A}_i^z)^2 - (\mathcal{A}_i^y)^2] + 2p_x\}$$
$$\times \{c_s[k_i^2 + (\mathcal{A}_i^z)^2 - (\mathcal{A}_i^y)^2] + 2p_y\} = 0, \quad (22)$$

and we come to the fourth-order algebraic equation for $k(\mathbf{r})$. It can be solved numerically, and a resulting dependence of $k_i(\mathbf{r})$ on the gauge field in the integral $\xi(\mathbf{r}) = \int_C k_i(\mathbf{r}) dr_i$ leads to the Berry phase acquired by the spin wave propagating along the contour C.

We can find the solution of Eq. (22) analytically in the limit of weak gauge potential $|\mathcal{A}_i^z| \ll k$, which corresponds to the adiabatic variation of the magnetization direction $\mathbf{n}(\mathbf{r})$ and also the adiabatic rotation in space of the elliptic trajectory, $|\partial \psi / \partial r_i| \ll k$. Then, in the first order of $\mathcal{A}_i^z$ we find

$$k_i(\mathbf{r}) \simeq \frac{g_i}{\sqrt{c_s}}[(\omega^2 + p^2)^{1/2} - p]^{1/2} + g_i g_j \mathcal{A}_j^z(\mathbf{r})(1 + p^2/\omega^2)^{-1/2},$$

(23)

where $p = |p_x - p_y|$.

Using Eq. (23) and taking the vector **g** along the tangent at each point of a closed contour C, we find the Berry phase

$$\gamma_B(C) = \oint_C \frac{\mathcal{A}_i^z(\mathbf{r}) dr_i}{(1 + p^2/\omega^2)^{1/2}}.$$

(24)

As follows from (24), the Berry phase $\gamma_B(C)$ in the anisotropic system acquires an additional factor $\kappa \equiv (1 + p^2/\omega^2)^{-1/2}$ depending on the magnetic anisotropy parameter p.

The denominator in (24) has a simple geometrical interpretation. Indeed, the coefficients a and d in the semiclassical solution (15) and (16) are the ellipse parameters, which are related to the anisotropy factor p

$$\frac{a}{d} = \left(1 + \frac{p^2}{\omega^2}\right)^{1/2} - \frac{p}{\omega}.$$

(25)

Correspondingly, we can relate the parameter $\kappa = (1 + p^2/\omega^2)^{-1/2}$ in Eq. (24) to the geometry parameters of the ellipse

$$\kappa = \sin 2\Theta,$$

(26)

where $\Theta = \arctan(d/a)$.

Using definition (7), the Berry phase finally can be presented as

$$\gamma_B(C) = \oint_C \left[(\kappa - 1)\frac{\partial(\psi + \phi)}{\partial r_i} + \kappa(\cos\theta - 1)\frac{\partial\phi}{\partial r_i}\right] dr_i.$$

(27)

In this expression we extracted a term proportional to $2\pi N$, $N \in \mathbf{Z}$. This allows us to avoid the multivaluedness of Berry phase in the absence of anisotropy when $\kappa = 1$.[23] The first term in (27) is proportional to the total winding number of rotations associated with the angles ψ and ϕ, whereas the second term is a spherical angle on S_2, which is the mapping space of the vector field $\mathbf{n}(\mathbf{r})$. The second term in (27) has a standard interpretation of the Berry phase as the magnetic flux penetrating the contour on S_2, when the field is created by a monopole at the center of Berry sphere. Following this idea, one can interpret the first term in (27) as the flux created by the magnetic string along the z axis, penetrating through the mapping contour on the unit circle.[23] In accordance with Eq. (27), this contribution to the Berry phase vanishes for isotropic magnetic systems, $\kappa = 1$. The first term in (27) is the *topological* Berry phase (it depends only on the winding number), in contrast to the *geometric* Berry phase of the second term in (27).[23]

As follows from (24), the effective gauge field for spin waves in the anisotropic system is $\widetilde{\mathcal{A}}_i = \kappa A_i^z$, and the corresponding topological field acting on the magnons can be calculated as the curvature of connection $\widetilde{\mathcal{A}}_i(\mathbf{r})$

$$B_i = \kappa \epsilon_{ijk} \frac{\partial \mathcal{A}_k^z}{\partial r_j} + \epsilon_{ijk} \frac{\partial \kappa}{\partial r_j} \mathcal{A}_k^z.$$

(28)

Note that there is a contribution related to the variation in space of the anisotropy parameters [second term in Eq. (28)].

We consider now in more detail the motion of elliptic spin wave in the adiabatic regime. The anisotropy suppresses one of the components s_x or s_y breaking the symmetry with respect to rotations around the z axis. Correspondingly, there is no gauge invariance $s_+ \to e^{i\varphi}s_+$ and $s_- \to e^{-i\varphi}s_-$ for the circular components, and the motion of magnetization in the spin wave is elliptical. In the adiabatic limit of $|\mathcal{A}_i^z| \ll k$, the solutions for s_x and s_y are given by Eqs. (15) and (16) with $b = f = 0$ and the ratio of amplitudes (a/d). Thus, we could expect the local invariance to transformations preserving the value of $s_x^2 + (d/a)^2 s_y^2 = $ const instead of simple rotations in the $x - y$ plane.

Using the Fourier transformation of Eqs. (18) and (19) for $s_{x,y}$, we find the following equation for the elliptic components of spin wave, $\widetilde{s}_\pm = s_x \pm i(d/a)s_y$:

$$\left[\omega + 2c_s k_i \mathcal{A}_i^z - \widetilde{c}_s\{k_i^2 + (\mathcal{A}_i^z)^2\} - \frac{pd}{2a}\right]\widetilde{s}_+$$
$$- \frac{a}{2d}\left[c_s\{(k_i^2 + (\mathcal{A}_i^z)^2)\left(1 - \frac{d^2}{a^2}\right) - \frac{pd^2}{a^2}\right]\widetilde{s}_- = 0,$$

(29)

and the complex conjugate to (29), where $\widetilde{c}_s = c_s(1 + d^2/a^2)a/2d$, and we determine the d/a from the condition of vanishing of the second bracket in Eq. (29). This condition determines the ellipticity factor, and we find that it coincides with Eq. (25) in the limit of $|\mathcal{A}_i^z| \ll k$. Thus, we come to the following equation for the elliptic wave in the gauge field:

$$\left\{\omega + 2c_s k_i \mathcal{A}_i^z - \widetilde{c}_s[k_i^2 + (\mathcal{A}_i^z)^2] - \frac{pd}{2a}\right\}\widetilde{s}_+ = 0.$$

(30)

This equation is not gauge invariant but in the adiabatic regime, neglecting the difference in small terms of the order of $(\mathcal{A}_i^z)^2$, we can present it as

$$\left[\omega - \widetilde{c}_s(k_i - \kappa \mathcal{A}_i^z)^2 - \frac{pd}{2a}\right]\widetilde{s}_+ = 0.$$

(31)

Equation (31) contains a factor $\kappa < 1$ before $\mathcal{A}_i^z$, and formally looks like the equation of motion of a particle moving in the reduced gauge field, which in turn leads to an effective suppression of the Berry phase. The calculation of Berry phase using Eq. (31) with the gauge field suppressed by factor κ leads us again to Eq. (24).

In the absence of anisotropy and in the adiabatic approximation, the solution of spin-wave equations has a simple form. The equations for circular components (12) are separated

$$\pm i\frac{\partial s_\pm^{(0)}}{\partial t} = -c_s\left(\frac{\partial}{\partial r_i} \mp i\mathcal{A}_i^z\right)^2 s_\pm^{(0)},$$
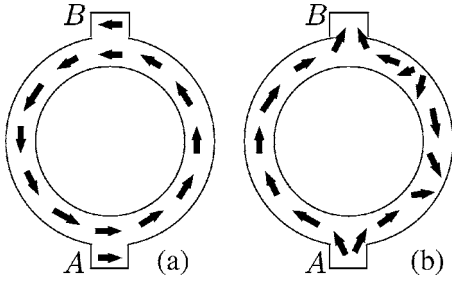
(32)

and the corresponding solution is

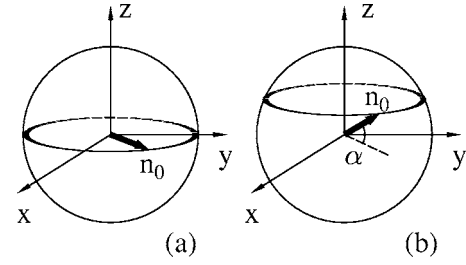FIG. 1. Two rings with topologically nontrivial magnetization field.

FIG. 2. Mapping of the ring to the **n**-space S_2 (red contour) in the case of in-plane vortex magnetization shown in Fig. 1(a) (a) and for the same geometry with magnetization vector deviating from the plane to the angle α (b). The Berry phase is 2π in case (a) and π for $\alpha=\pi/6$ in case (b).

$$s_+(\mathbf{r},t) \sim \exp\left[i\int k_i(\mathbf{r})dr_i - i\omega t\right], \qquad (33)$$

with $k_i(\mathbf{r}) \simeq g_i\omega/c_s^{1/2} + g_ig_j\mathcal{A}_j^z(\mathbf{r})$. The spin wave propagating along a closed contour C acquires the Berry phase of Eq. (24) with $p=0$.

Using Eqs. (7) we can present the topological field (28) in the absence of anisotropy as

$$B_i = -\sin\theta\,\epsilon_{ijk}\frac{\partial\theta}{\partial r_j}\frac{\partial\phi}{\partial r_k}. \qquad (34)$$

It does not depend on the angle ψ, related to the choice of gauge as in the case of electromagnetism.

By creating a certain metastable configuration of the magnetization $\mathbf{n}_0(\mathbf{r})$ in the ferromagnet, we simulate an effective gauge potential $\widetilde{\mathcal{A}}_i(\mathbf{r})$, acting on the spin waves similar to the magnetic field in case of electrons. In particular, when the averaged in space topological field (28) is not zero, there arises the Landau quantization of the energy spectrum of magnons. In the absence of anisotropy, we find the quantized spectrum $\hbar\omega_n = 2\hbar c_s|\langle\mathbf{B}\rangle|(n+1/2)$, where $\langle\ldots\rangle$ means the average in space.

IV. INTERFERENCE OF SPIN WAVES IN MESOSCOPIC RINGS

Let us consider now the ring geometry of a ferromagnet with a topologically nontrivial metastable magnetization $\mathbf{n}_0(\mathbf{r})$. It can be, for example, a magnetization vortex [Fig. 1(a)] or an even number of domain walls in one branch of the ring as presented in Fig. 1(b). Such a magnetization profile presents a metastable magnetic state.

Let us consider first the case when there is no anisotropy. If $kL \ll 1$ (adiabatic regime), the low-energy magnetic excitations of the metastable state are described by Eq. (32). Due to the presence of gauge potential $\mathcal{A}_i^z(\mathbf{r})$, there is a phase shift of waves propagating from the point A, where the waves are excited, to the observation point B (see Fig. 1). The phase shift (Berry phase) equals the integral $\oint\mathcal{A}_i^z(\mathbf{r})dr_i$ along the ring, and by using the Stokes theorem can be calculated as the flux Φ of topological field B defined in Eq. (34). It can also be presented as the spherical angle enclosed by the mapping of the ring to the circle at the unit sphere S_2. This way we can find the phase shift of 2π and πN_d for Figs. 1(a) and 1(b), respectively, where an even N_d is the number

of domain walls in the right arm of the ring. For example, in the case of Fig. 1(b) with two domain walls in the right arm, there is no interference of spin waves excited in A and coming to the point B because the corresponding phase shift is 2π.

In the absence of anisotropy, the interference in the ring can be induced by rotating all magnetic moments from the plane to a certain angle α [the corresponding mapping is presented in Fig. 2(b)]. The Berry phase associated with the path along the ring will be smaller than 2π. For $\alpha=\pi/6$ the Berry phase turns out to be π. This means that the experiment with interference of spin waves propagating from A to B through two different arms of the ring would result in a complete suppression of the outgoing from B spin wave. Physically, it can be realized using the ring with very small easy-plane anisotropy, $p/\omega \ll 1$, in a weak external magnetic field along z axis.

A similar idea was recently proposed by Schütz *et al.*[24] for the radial orientation of magnetic moments under inhomogeneous magnetic field directed from some point at the axis of the ring. This magnetic field creates a "crown" of magnetic moments, and the corresponding mapping is similar to that presented in Fig. 2(b).

However, in the case of nonvanishing easy-plane anisotropy, there is no need to apply magnetic field to provide the interference of spin waves propagating in the geometry of Fig. 1(b). In this case the Berry phase is given by Eq. (27) with $p=\gamma\lambda/2M_0=$const, and we obtain the difference in phases for two waves $\Delta\gamma_B = \pi N_d/(1+p^2/\omega^2)^{1/2}$. Thus, the interference of spin waves should be clearly seen for the two-arm geometry with domain walls. The computer simulation experiments[17] confirm this expectation.

Another possibility to observe the interference of spin waves can be presented by the geometry of a wide ring (thin-wall cylinder) as presented in Fig. 3. Assuming the easy-plane anisotropy of the ribbon, we obtain the ground state with a homogeneous magnetization along the axis of cylinder. The anisotropy axis is oriented radially in each point of the cylinder, and the corresponding local frame is shown in the figure as $x'y'z'$. Due to the homogeneous magnetization, we get $\theta=0$, and from (7) we obtain $\mathcal{A}_i^z(\mathbf{r}) = \partial(\psi+\phi)/\partial r_i$. The components of anisotropy vector are $u_x=\cos\varphi$ and $u_y=\sin\varphi$. Using Eq. (34), we find that the condition $\widetilde{u}_y=0$ reduces to $\psi+\phi=\varphi$, and from (34) we obtain $\widetilde{u}_x=1$. Corre-
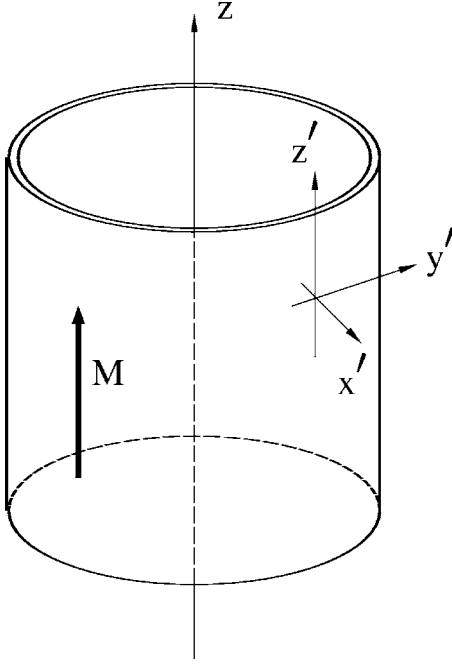
FIG. 3. Spin waves on a wide ring (thin-wall cylinder made of easy-plane ferromagnetic ribbon). The magnetization is along axis z.

spondingly, the anisotropy parameter $p = \gamma\lambda/2M_0$, and we obtain the Berry phase for the closed contour on the ring $\gamma_B(C) = 2\pi N_C/(1 + p^2/\omega^2)^{1/2}$, were N_C is the winding number of the contour C.

The Berry phase of the spin wave propagating in magnetic ring plays the similar role as the phase of electron wave function in the Aharonov-Bohm effect with magnetic flux penetrating through the ring. A string through the ring plays the role of the magnetic flux.[23] However, the flux created by the string does not depend on the size or shape of the magnetic ring. Correspondingly, the Berry phase associated with the string has the topological origin, which makes it different from the Aharonov-Bohm effect induced by the magnetic-field flux thorough the conductive ring.

V. SPECTRUM OF MAGNONS IN A RING WITH UNIAXIAL ANISOTROPY

The other example is a ring with uniaxial anisotropy in a homogeneous magnetic field **B** along the axis z as presented in Fig. 4. Due to the anisotropy and exchange interaction, the magnetization along the ring is oriented as in Fig. 4, creating a certain angle θ out of the ring plane.

We take the anisotropy function $\mathcal{F}$ in the form

$$\mathcal{F}\{\mathbf{n}\} = -\frac{\lambda}{2}n_\varphi^2. \qquad (35)$$

which corresponds to the uniaxial anisotropy along the ring. The local frame is chosen with the axis z along the magnetization at each point, and with the z–x plane tangential to the ring (parallel to the axis z). In this case we find $\widetilde{\mathcal{F}} = -\lambda s_x^2 \cos^2\theta/2$. The angles determining the orientation of local frame are $\psi = 0$ and $\phi = r/R$.
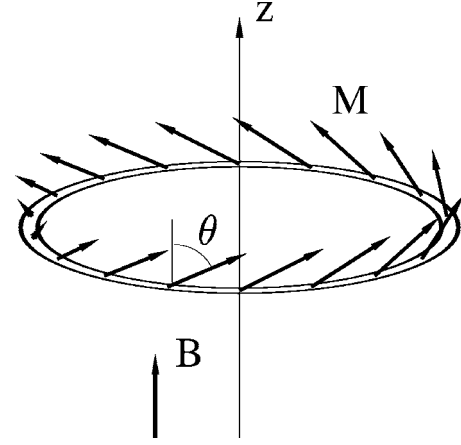

FIG. 4. Magnetization in a ring with uniaxial magnetic anisotropy in a homogeneous external magnetic field along the z axis.

We can calculate the angle θ using Eqs. (1) and (32) with vectors **u** and **B** along the axis z. Then, using the polar coordinates and the condition that n_φ and n_z do not depend on the point along the ring, we find the energy

$$E = 2\pi\zeta_0 R\left(\frac{an_\varphi^2}{2R^2} + \frac{\lambda n_z^2}{2} - M_0 Bn_z\right). \qquad (36)$$

Substituting $n_\varphi = \sin\theta$, $n_z = \cos\theta$, we calculate the angle θ minimizing the energy (36)

$$\cos\theta = \min\left\{1, \frac{M_0 B}{\lambda - a/R^2}\right\}, \qquad (37)$$

for $\lambda > a/R^2$, and $\theta = 0$ for $\lambda \leq a/R^2$.

The gauge potential in magnetic ring induces the energy splitting of magnons propagating in the opposite directions.[23] Using Eqs. (7) we find $\mathcal{A}^x = \sin\theta/R$, $\mathcal{A}^y = 0$, and $\mathcal{A}^z = \cos\theta$. It should be noted that the gauge potential is constant along the ring, so there is no need to use the adiabatic approximation to determine the energy spectrum of magnons.

Using Eqs. (10) and (11), after Fourier transformation over time t and coordinate r along the ring, we find

$$\omega s_x = ic_s[k_n^2 s_y + 2ik_n\mathcal{A}^z s_x + (\mathcal{A}^z)^2 s_y + (\mathcal{A}^x)^2 s_y] + i\Delta_0 s_y,$$

$$\omega s_y = -ic_s[k_n^2 s_x - 2ik_n\mathcal{A}^z s_y + (\mathcal{A}^z)^2 s_x] + ips_x - i\Delta_0 s_x, \qquad (38)$$

where $\Delta_0 = \gamma B$. Here, the momentum takes discrete values $k_n = 2\pi n/L$ ($n = 0, \pm 1, \ldots$) to provide the periodicity of solution for the spin wave along the ring. We can use (38) to find the equation for elliptic components of the spin wave

$$2(\omega + 2c_s k_n \mathcal{A}_z)\widetilde{s}_+$$

$$= \left\{[c_s k_n^2 + c_s(\mathcal{A}^z)^2 + \Delta_0]\frac{\eta^2 + 1}{\eta} + \frac{c_s(\mathcal{A}^x)^2}{\eta} - \eta p\right\}\widetilde{s}_+$$

$$+ \left\{[c_s k_n^2 + c_s(\mathcal{A}^z)^2 + \Delta_0]\frac{\eta^2 - 1}{\eta} - \frac{c_s(\mathcal{A}^x)^2}{\eta} - \eta p\right\}\widetilde{s}_-, \qquad (39)$$
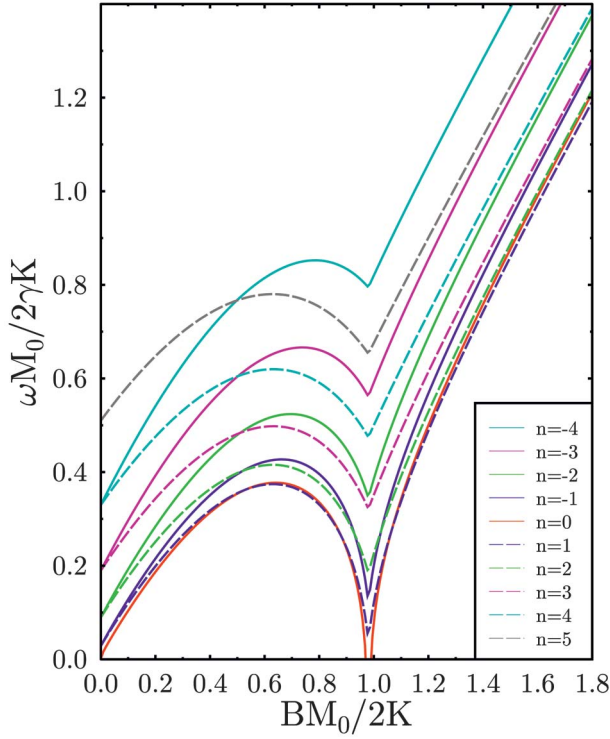
FIG. 5. (Color) Energy spectrum of magnons in the ring (Fig. 4) as a function of external magnetic field B, where $K = \lambda/2$.

where $\eta = d/a$ is the ellipticity. The expression in the second curved brackets vanishes for

$$\eta^2 = \frac{c_s k_n^2 + c_s (\mathcal{A}^z)^2 + c_s (\mathcal{A}^x)^2 + \Delta_0}{c_s k_n^2 + c_s (\mathcal{A}^z)^2 + \Delta_0 - p}. \qquad (40)$$

Then, using (39) and (40) we find the energy spectrum of magnons in the ring

$$\omega_n = \tilde{c}_s \left(k_n - \frac{2\eta}{1+\eta^2} \mathcal{A}^z \right)^2 + \frac{c_s}{2} \left(1 - \frac{4\eta}{1+\eta^2} \right) (\mathcal{A}^z)^2$$

$$+ \frac{c_s}{2\eta} (\mathcal{A}^x)^2 - \frac{\eta p}{2} + \Delta_0 \frac{1+\eta^2}{2\eta}. \qquad (41)$$

The spectrum is shown in Fig. 5 for several first values of n as a function of the magnitude of field B. We take the parameters $\lambda/2 = \pi M_0^2$ (which corresponds to the dipolar shape anisotropy of a magnetic cylinder), $R = 75$ nm, and $\lambda R^2/a = 200$. All the curves have a critical point B_c corresponding to the magnetic field, for which the magnetization **M** starts to deviate from the direction with $\theta = 0$. In view of Eq. (37), $B_c M_0 = \lambda - a/R^2$. For $n = 0$ the energy of magnons at this point is the soft mode with $\omega = 0$. This mode corresponds to a uniform rotation of spins at each point of the ring toward a tangential to the ring direction. In the local frame it is the uniform rotation corresponding to the state with $n = 0$.

For $n \neq 0$ and $B \rightarrow 0$ we can find the spectrum in linear B approximation

$$\omega_n(B) \simeq \omega_n(0) + \gamma B \left(1 + \frac{1}{4|n|(1+n^2)^{3/2}} \right) - \frac{2c_s nB}{\lambda R^2 - a}, \qquad (42)$$

where $\omega_n(0) = \omega_{-n}(0)$. This shows that the spectrum is degenerate at $B = 0$ with respect to the sign of n. The splitting for $B \neq 0$ in the linear approximation gives the curve for positive n below the one for the same negative in accordance with Fig. 5.

At large magnetic field B, the systematics of levels should be changed. Namely, the lowest energy mode corresponds to the uniform *global* deviation of orientation of all spins from the direction along the z axis. In the local frame, it corresponds to the mode with $n = 1$. Thus, it would be natural to label the modes with index $m = n - 1$, so that the lowest in energy is the spin wave with $m = 0$. In the limit of $B \rightarrow \infty$, each pair of modes $\pm m$ is degenerate. It is clearly seen from Eq. (41) with $\eta = 1$. The splitting of these modes for $B > B_c$ demonstrates the existence of *topological* Berry phase for the magnons on the ring[23] because the equilibrium state is the homogeneous magnetization, which leads to the vanishing of *geometric* Berry phase.

VI. CONCLUSIONS

We calculated the Berry phase associated with the propagation of magnons in inhomogeneous ferromagnets and mesoscopic structures with topologically nontrivial magnetization profile. We found that the most important effect is related to the magnetic anisotropy. Due to the anisotropy, the Berry phase for magnons is lower than a standard value of the spherical angle on the Berry sphere with a monopole. Besides, an additional contribution to the Berry phase arises in anisotropic systems, which can be viewed as an effect of the gauge string penetrating through the mapping contour on the unit circle.

Using these results, we demonstrated that the Berry phase can be observable in interference experiments on magnetic rings, and is controllable using the additional homogeneous magnetic field.

The spin-wave equations in the quasi-one-dimensional case of mesoscopic rings are similar to the classical theory of anisotropic 1D ferromagnets. However, as we show in Sec. IV and V, the real geometry of rings determines the geometric and topological Berry phase of magnons, thus leading to effects like quantization of the magnon energy spectrum.

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