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On the maximum of the field-dependent susceptibility in ferromagnetic materials

Y.T. Millev [a,b,*], M. Fähnle [a], M.A.B. Whitaker [c]

[a] *Institut für Physik, MPI für Metallforschung, Heisenbergstr. 1, D-70569 Stuttgart, Germany*
[b] *Institute of Applied Physics, Technical University, 1756 Sofia, Bulgaria*
[c] *Department of Pure and Applied Physics, Queen's University of Belfast, Belfast BT7 1NN, Northern Ireland, UK*

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Abstract

We propose two new methods to describe the ferromagnetic field-dependent susceptibility within the mean-field theory. A *parametric* approach valid for any value of temperature, applied field, and spin quantum number is developed; within this approach, the scaling functions for magnetization and susceptibility are determined for values of the reduced field, smaller than 10^{-3}. A simple *analytic* derivation of the scaling functions is also given. As the susceptibility maximum is found to occur at a value of the relevant scaling variable which is of the order of unity, it cannot be accurately described by series expansions of the scaling function. A nonlocal parabolic approximant to the scaling function is constructed which reproduces its main features exactly. The methods of this paper are relevant to the study of the field-dependent susceptibility of any ferromagnet in which long-range forces are known to dominate. It is suggested that the analysis be tested on the examples of the 'mean-field' ferromagnets $HoRh_4B_4$ and $ZrZn_2$. The whole scheme should be regarded as contributing to the elaboration of the advantageous procedure for the determination of two independent critical exponents, which is based on general scaling analysis for the field-dependent susceptibility and which avoids painstaking measurements of the exact Curie temperature.

1. Scaling theory predictions for the field-dependent susceptibility of a ferromagnet

According to scaling theory, the thermodynamic potential of a ferromagnet close to its critical point is a generalized homogeneous function of the relevant thermodynamic variables [1,2]. As a corollary, one obtains for the magnetization the scaling form

$$m(t,h) = |\tau|^{\beta} \cdot g_{\pm}\left(\frac{h}{|\tau|^{\beta\cdot\delta}}\right) = |\tau|^{\beta} \cdot g_{\pm}(s), \qquad (1)$$

where m and h are the reduced magnetization and applied magnetic field, respectively, while $\tau \equiv (T - T_C)/T_C$, with T_C being the Curie temperature; $s = h/|\tau|^{\beta\cdot\delta}$ is the scaling variable for the scaling function g. The latter has two branches, g_+ and g_-, corresponding to temperatures above and below T_C; β and δ are the critical exponents for the spontaneous magnetization [1]. Alternatively, the scaling form for the magnetization can also be given as

$$m(t,h) = h^{1/\delta} \cdot f_{\pm}(\sigma), \qquad (2)$$

where $f_{\pm}(\sigma) = \sigma^{\beta} \cdot g_{\pm}(1/\sigma^{\beta\delta})$ and $\sigma = \tau/h^{1/\beta\delta}$. The corresponding equivalent scaling forms for the

* Corresponding author. Fax: +49-711-689-1010; email: millev@physix.mpi-stuttgart.mpg.de.

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susceptibility derive from its definition and Eqs. (1) and (2):

$$\chi \equiv \left(\frac{\partial m}{\partial h}\right)_T = |\tau|^{-\gamma} \cdot G_{\pm}(s) = h^{1/\delta-1} \cdot F_{\pm}(\sigma), \tag{3}$$

where γ is the critical exponent for the susceptibility, while the functions $G_{\pm}(s)$ and $F_{\pm}(\sigma)$ can be easily obtained from $g_{\pm}(s)$ and $f_{\pm}(\sigma)$, respectively, via straightforward differentiation. In particular,

$$F_{\pm}(\sigma) = \frac{1}{\delta}\{f_{\pm}(\sigma) - 1/\beta \cdot \sigma \cdot f'_{\pm}(\sigma)\}. \tag{4}$$

In deriving Eq. (3), use was made of Widom's relation, $\gamma = \beta(\delta - 1)$.

In zero applied field, χ diverges at T_C, i.e. at $\tau = 0$. In a finite field, this divergence is smeared to a maximum χ_m of finite height, which is located at some temperature T_m above T_C. If the field is increased, χ_m falls off while T_m increases. Both quantities scale with applied field, but with different exponents.

One way to see how this scaling arises is to simply consider the necessary condition for the existence of a maximum of $\chi(\tau,h)$ in one of the equivalent forms. Working with $T > T_C$ and neglecting the subscript '+' of $\phi(s)$, it follows that

$$\frac{\partial \chi}{\partial \tau} = h^{(1/\delta)-1-(1/\beta\delta)} \cdot F'(\sigma), \tag{5}$$

and, in finite fields, this can only be zero for $F'(\sigma) = 0$ or, equivalently, for

$$\sigma \cdot f''(\sigma) + (1-\beta) \cdot f'(\sigma) = 0. \tag{6}$$

The simple and crucial observation is that the terms on the left-hand side in Eq. (6) are functions of the scaling variable σ only, and not of its constituent variables separately. The maximum is then at $\sigma_m = \tau_m / h^{1/\beta\delta}$, i.e. at

$$\tau_m = \sigma_m \cdot h^{1/\beta\delta}, \tag{7}$$

where σ_m is the solution of Eq. (6). Inserting τ_m into Eq. (3), one gets

$$\chi_m(h) \equiv \chi(\tau_m, h) = h^{1/\delta-1} \cdot F(\sigma_m). \tag{8}$$

In other words, one obtains that the maximum of the susceptibility χ_m and its location $\tau_m = (T_m - T_C)/T_C$ scale with field as

$$\chi_m \sim h^{1/\delta-1}, \tag{9}$$

$$\tau_m \sim h^{1/\beta\delta}. \tag{10}$$

The usefulness of, and the advantages offered by, these scaling relations have been discussed and exploited in a number of studies [3–7]. The relations (9)–(10) imply, for instance, that by measuring the well-pronounced experimental feature (the maximum of the susceptibility) and its location, one can determine the values of *two* scaling, exponents β and δ, and then all the other critical exponents can be derived by virtue of simple scaling relations [1,2]. Moreover, one does not need to know the value of T_C to determine β and δ, and it is well known that other types of scaling fits depend crucially on the experimental estimate of T_C [8,9].

Furthermore, it has been observed [10] that a plot of the susceptibility (measured in a number of fixed fields as a function of temperature and normalized to its peak value) against the scaling variable $\sigma = \tau / h^{1/\beta\delta}$, should help construct the normalized scaling function $\bar{F}(\sigma)$:

$$\bar{F}(\sigma) = \frac{\chi(\tau,h)}{\chi(\tau_m,h)} = \frac{F(\sigma)}{F(\sigma_m)}. \tag{11}$$

In a later section, we construct this function in the mean-field (MF) approximation by using two different *and* new approaches.

It must be emphasized here that the ease with which the scaling relations for the maximum of the susceptibility are derived in the general, unspecified case is rather delusive and is in a way 'inversely proportional' to the care which must be exercised in a specific case such as the MF approximation. The point is that straightforward expansions of the scaling functions for small values of their respective arguments are often considered as trivial and plausible because of the analyticity of the scaling functions, the singularities being factored out as in Eqs. (3). However, if we consider for instance the variable σ and its related functions $f(\sigma)$ and $F(\sigma)$, at the maximum of the susceptibility one has

$$\sigma = \sigma_m = \tau_m / h^{1/\beta\delta} \sim O(h^0), \tag{12}$$

which is what follows by virtue of Eq. (10). Hence, a perturbative treatment of the scaling functions $f(\sigma)$

and $F(\sigma)$ cannot be justified if one is interested, as we are, in a feature occurring at values of the expansion variable of the order of unity. Even if the existence of the extremum is not destroyed during such a procedure, its location is certainly quite inexact. Besides, one cannot rely on accidental smallness of σ_m stemming, eventually, from the spin factors involved. We find that σ_m is not small for any S in the MF case (cf. Eq. 35).

2. Mean-field analysis of the field-dependent susceptibility

While the implications of the above scaling analysis for the experimental study of different ferromagnetic materials have been appreciated and successfully implemented [3–5], the theoretical computation of $g(s)$ or $f(\sigma)$ and, consequently, of $G(s)$ or $F(\sigma)$ for any nontrivial interacting model is still an open problem. To our knowledge, only the somewhat academic, though instructive, case of a paramagnet with a conditional phase transition at $T_C = 0$ [11] and the MF analysis for χ_m and τ_m as functions of h for spin $\frac{1}{2}$ [7] have been tackled.

It is therefore necessary to study in sufficient detail and rigor the maximum of the susceptibility χ_m and its location τ_m as functions of applied field for any spin quantum number S in the MF approximation. On the one hand, this would fill a gap in the theoretical understanding of these features of criticality by using a model of *interacting* magnetic moments. In fact, this model describes exactly some exemplary cases of 'mean-field' ferromagnets such as $HoRh_4B_4$ [12], where, presumably, the interactions responsible for the ferromagnetic ordering are of long range. It would be of great value to carry out the experimental measurements for χ_m, τ_m, and $g(s)$ or $f(\sigma)$ in this and similar materials. On the other hand, the results obtained below provide for a delineation of the scaling regime in the presence of an external field, a problem which has not been satisfactorily treated so far, despite of its great theoretical and experimental significance [13,14].

In the MF approximation, one thinks of any individual magnetic moment as being immersed in the averaged effective field created by all the other moments plus the applied field. The entangled many-body interactions are thus reduced to the basic dipole interaction of a moment with a field. Calculating the average magnetic moment by standard statistical mechanical means leads to a self-consistent equation for the magnetization (Eq. 13). Solving the self-consistent equation would give, in principle, the temperature and external field dependence of magnetization and, consequently, of susceptibility and of other magnetization-related quantities.

The well-known self-consistent equation for the FM magnetization per magnetic site of spin S is

$$m = B_S(x), \tag{13}$$

where

$$B_S = a\coth(ax) - b\coth(bx) \tag{14}$$

is the Brillouin function with $a = (2S+1)/2S$ and $b = 1/2S$, while

$$x = \frac{cm + h}{t} \tag{15}$$

is the generalized effective field [15]; $c = 3S/(S+1)$ is another spin-dependent constant. In contrast with the qualitative discussion of the scaling relations, we need to specify the natural reduced quantities: $m = M/M_0$ with the magnetic moment M of the system and its saturation value $M_0 = Ng_s\mu_B S$ (g_s is the spin-only Landé factor and μ_B is the Bohr magneton, while N is the number of magnetic ions), $t = T/T_C$ is the reduced temperature; and $h = g_s\mu_B S B_0/k_B T_C$ is the reduced external magnetic field in which B_0 is the flux density of the applied field and k_B is Boltzmann's constant. Note the difference between the temperature variables t and τ: $t = 1 + \tau$.

The difference between relations (1) and (13) is, of course, that the first one gives the *function* $m(t,h)$, though with an unknown scaling function involved, while the latter is the *equation* which has to be solved to find $m(t,h)$ for all t and h. Besides, although the second relation is *not* a scaling one and is therefore applicable for any t and h within the MF approximation, it may provide information about the MF scaling region which is characterized by the set of MF critical exponents [1].

With the definition (3), and by carrying out the necessary chain differentiation in Eq. (13), one finds for the MF differential susceptibility

$$\chi \equiv \frac{\partial m}{\partial h} = \left(\frac{t}{B_S'(x)} - c\right)^{-1}, \tag{16}$$

where $B'_S(x)$ is the derivative of $B_S(x)$ with respect to x.

One may easily derive the asymptotic behavior of m and χ on the critical isotherm ($T = T_C$). By Eqs. (13) and (16), one obtains

$$m_c \equiv m(\tau = 0, h) = m_0 \cdot h^{1/\delta} \quad (h \to 0). \tag{17}$$

In the MF approximation, one finds $\delta = 3$ and

$$m_0 = \frac{1}{c}\left(\frac{15}{a^2 + b^2}\right)^{1/3} h^{1/3}. \tag{18}$$

Similarly for $\chi(\tau = 0, h)$,

$$\chi(\tau = 0, h) \to \frac{1}{c}\left[\frac{5}{9(a^2 + b^2)}\right]^{1/3} h^{-2/3} \quad (h \to 0). \tag{19}$$

For arbitrary τ and h, the standard methods of solving Eq. (13) involve a numerical solution or a graphical procedure. Both of these have proven cumbersome enough; in particular, this seems to explain why the problem with the MF field-dependent susceptibility has not been studied exhaustively until now. A particularly simple and effective *parametric* realization of the MF objectives in computing the temperature and field dependences of static susceptibility has been suggested and implemented for both ferro- and antiferromagnetic cases [16], whereby use was made of the physically most appealing version from among a variety of parametrizing opportunities [17]. The salient features of the parametric approach amount to expressing all relevant physical quantities as explicit functions of a single parameter. The parameter is then allowed to sweep the range of its physically allowed values and the quantities of interest are computed independently. The results for whatever dependence might be interesting are obtained by collecting pairs of points corresponding to the same value of the flowing parameter. The output can subsequently be cast into a tabular or graphical form. The suitable choice for a flow parameter was found to be the generalized effective field x which, by its definition (15), sweeps between zero and infinity.

Both cases of *fixed temperature* or *fixed external field* can be treated equally easily with $m = m(x)$ and $\chi = \chi(x)$ being explicit functions of x. For fixed t, one uses

$$h = t \cdot x - c \cdot m(x) = h(x), \tag{20}$$

with t and S as known input parameters, while for fixed h

$$t = \frac{c \cdot m(x) + h}{x} = t(x), \tag{21}$$

with h and S as input parameters. One possible way to carry out the study of χ_m and t_m as functions of applied field would be to collect the corresponding data from a sufficient number of plots for different values of h in order to construct the said field dependence in a tabular and, consequently, graphical form. One could, however, do much better than that.

3. Novel parametric approach to study the maximum of the field-dependent susceptibility in the MF approximation

Some experience with the proposed parametric approach [16,17] and the possibility for its extension to treat highly non-trivial problems, such as the temperature dependence of magnetic anisotropy in cubic ferromagnets within the random-phase approximation [18], has led us to another extension of the scheme in order to study the field dependence of χ_m and t_m. The idea is that one could avoid bookkeeping activities for collecting sets of data from separate parametric sweeps; instead, one could derive and scrutinize the MF equations for t_m and χ_m to find a shortcut. From the condition $(\partial\chi/\partial t)_h = 0$ and with the expression for $\chi(t, x(t,h))$ already given in Eq. (16), one finds that the following equation must be satisfied at the point t_m of maximal susceptibility:

$$B'_S(x) - \frac{B''_S \cdot x \cdot t}{c \cdot B'_S(x) - t} = 0. \tag{22}$$

The lucky circumstance is that this equation can be solved to find t_m and, hence, χ_m as functions of $x_m \equiv x_{|t=t_m}$ explicitly:

$$t_m = 1 + \tau_m = \frac{c[B'_S(x_m)]^2}{B'_S(x_m) + x_m \cdot B''_S(x_m)}; \tag{23}$$

$$\chi_m \equiv \chi(t = t_m, h) = \left(\frac{t_m}{B'_S(x_m)} - c\right)^{-1}. \tag{24}$$

The crucial observation is that one can get $\chi_m(h)$ and $t_m(h)$ by using the parametric method with x_m

as the sweeping parameter, thus compensating for, and indeed making use of, our ignorance of

$$x_{\mathrm{m}} = \frac{c \cdot \overline{m}(t_{\mathrm{m}},h) + h}{t_{\mathrm{m}}}, \tag{25}$$

where $\overline{m}$ is the value of the magnetization at $t = t_{\mathrm{m}}$. To carry out this scheme, one needs also the variable h as a function of the prospective sweeping parameter x_{m}. By Eq. (20) taken at $t = t_{\mathrm{m}}$,

$$h = x_{\mathrm{m}} \cdot t_{\mathrm{m}} - c \cdot \overline{m} = x_{\mathrm{m}} \cdot t_{\mathrm{m}}(x_{\mathrm{m}}) - c \cdot B_S(x_{\mathrm{m}}). \tag{26}$$

Eqs. (23)–(26) provide the basis for the parametric solution of the problem. Indeed, these equations amount to *explicit* expressions for

$$\chi_{\mathrm{m}} = \chi_{\mathrm{m}}(x_{\mathrm{m}}), \tag{27}$$

$$t_{\mathrm{m}} = t_{\mathrm{m}}(x_{\mathrm{m}}), \tag{28}$$

$$h = h(x_{\mathrm{m}}). \tag{29}$$

Letting x_{m} sweep between zero and infinity, one collects pairs of points to plot or tabulate $\chi_{\mathrm{m}}(h)$ and $t_{\mathrm{m}}(h)$ for any given value of S. The identification of a suitable sweeping parameter lies at the heart of the proposed solution. It must be emphasized that the parametric approach described in principle above effectively reduces the problem with solving the self-consistent equation for $m(t,h)$ to a purely computational procedure with explicitly known functions. Thus the method overrides the implicit character of Eq. (13) and allows one to proceed as if one has an explicitly known function $m(t,h)$.

4. Analysis of the MF scaling regime for the susceptibility in applied field

4.1. χ_m and τ_m as functions of field

The results from the application of the method are easily obtained for $\chi_{\mathrm{m}}(h)$ and $\tau_{\mathrm{m}}(h) = 1 - t_{\mathrm{m}}(h)$ for various values of spin. In order to uncover the expected MF scaling behavior and the domain of its validity, double logarithmic plots have been used in both cases. The dependence $\chi_{\mathrm{m}}(h)$ for any given S represents the crossover line between the low- and high-temperature domains [19]. The straight parallel parts for small $h < 10^{-3}$ in the plots (which are easy to generate and are not given here for the sake of brevity) indicate the existence of a region that extends over many decades of values of h, where χ_{m} and τ_{m} scale with some power of h. With slopes of $(-2/3)$ and $(2/3)$ for any S, one concludes unambiguously that there $\chi_{\mathrm{m}} \sim h^{-2/3}$ and $\tau_{\mathrm{m}} \sim h^{2/3}$. This region is thus the domain where MF scaling holds, in agreement with Eqs. (9) and (10), and with the MF scaling exponents $\beta = 1/2$ and $\delta = 3$.

At this stage, some further information about the maximum of the susceptibility in small fields can be worked out easily. With the realization that the maximum has its origin in the critical divergence of χ at $T = T_{\mathrm{C}}$ in zero field, one concludes that in small fields χ_{m} must occur for $T_{\mathrm{m}} \to T_{\mathrm{C}}^{+}$. In other words, $\tau_{\mathrm{m}} = (T_{\mathrm{m}} - T_{\mathrm{C}})/T_{\mathrm{C}} \to 0$ for $h \to 0$. This means that $x_{\mathrm{m}} = (c\overline{m} + h)/(1 + \tau_{\mathrm{m}}) \to 0$ as well, since $\overline{m} = m(\tau_{\mathrm{m}})$ also tends to zero in this regime. By Eqs. (23) and (24), one easily finds that

$$\chi_{\mathrm{m}} \to \frac{5}{2c(a^2 + b^2)} \frac{1}{x_{\mathrm{m}}^2}, \tag{30}$$

and

$$\tau_{\mathrm{m}} \to \tfrac{1}{5}(a^2 + b^2)x_{\mathrm{m}}^2, \tag{31}$$

whereby

$$\chi_{\mathrm{m}} \cdot \tau_{\mathrm{m}} \to \frac{1}{2c} = \frac{S+1}{6S} \quad (h \to 0). \tag{32}$$

This is an interesting result which complies with the general predictions of scaling theory, for it follows from Eqs. (7) and (8) that the product $\chi_{\mathrm{m}} \cdot \tau_{\mathrm{m}}$, when computed with the MF critical exponent $\gamma = 1$, and provided that the Widom relation holds, regardless of the values of β and δ, must be asymptotically constant in h. Interestingly, the value of this constant is precisely equal to the value of the maximum of the MF antiferromagnetic susceptibility in small applied fields [16]. In Fig. 1, we give the product $\chi_{\mathrm{m}} \cdot \tau_{\mathrm{m}}$, normalized against its asymptotic value of $\frac{1}{2}c$, as a function of h for various values of spin. To cover a larger variation of field, the product is plotted versus $\log(h)$.

For (unphysically) large h, one finds once again scaling of χ_{m} and τ_{m} with field, this time with slopes of (-1) and $(+1)$, respectively. Thus one

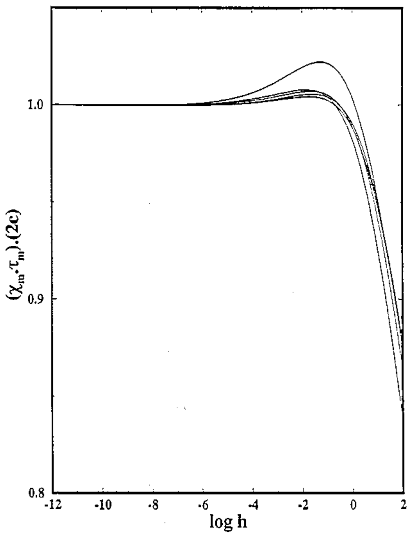

Fig. 1. The product $\chi_m \cdot \tau_m$ for different values of spin S normalized against $\frac{1}{2}c = (S+1)/6S$ (cf. Eq. 32). For $h \to 0$, it is unity. The onset of deviations from the MF scaling predictions with increasing field is easily observable.

detects the paramagnetic scaling regime described in Ref. [11], which has been characterized by the pathological critical exponents $\delta = \infty$ and $\beta = 1/\delta = 0$. These values of the 'paramagnetic' exponents comply with the slopes we find in this high-field regime. In the intermediate region $h \sim (10^{-3}–10^{1})$, no scaling with a simple power-law dependence in h is found.

4.2. Scope of validity of MF scaling in an applied field

There is further important information that can be deduced from the implementation of the proposed parametric method for the calculation of the field dependences of χ_m and τ_m. It concerns the width of the MF scaling region in the presence of an applied magnetic field. It is clear that the MF scaling region extends as far as the linear regime for small h in the double logarithmic plot. It is very easy to extract from the corresponding plots the values of h_c which limit the linear scaling dependence for any given S. One finds that $h_c \approx 10^{-3}$. To see what this estimate means physically, we restore the physical quantities according to the specification following Eq. (15). Inserting the values of the fundamental constants involved, one gets for the limiting value B_{0c} of the induction of the applied field

$$B_{0c} = (1.5)\frac{T_C}{g \cdot S} h_c \approx (1.5)\frac{T_C}{g \cdot S} \times 10^{-3}. \qquad (33)$$

It is interesting to see, within the framework of the MF theory, whether the result (33) is a specific outcome of the particular quantities studied. With the help of the parametric method, one is in the position to check, for instance, the state of matters with the MF scaling as seen on the critical isotherm. In this way one can compare the extent of the scaling region in external field as judged from the calculation of different quantities. To this end, one applies the 'old' parametric approach as described above with the general effective field x as a sweeping parameter [18] and computes $m(h)$. It is then found that the MF scaling region $m_c \approx h^{1/3}$ once again extends as far as $h_c \approx 10^{-3}$.This suggests that the extent of the MF scaling region in applied field is virtually insensitive to the choice of the field-dependent quantity which would be referred to when estimating the said region. This is in contrast with the estimates for the *fluctuation-dominated* region, which have been shown to be sensitive to the choice of the reference quantity [13,14].

There is an important point which has to be elucidated here. The perturbative estimates in Refs. [13,14] are based on the idea underlying the derivation of the Ginzburg–Levanyuk criterion [20], namely, that the MF approximation breaks down when the system is so close to the critical point ($T = T_C$, $H = 0$) that the fluctuation contribution to a given relevant quantity (susceptibility, specific heat, etc.) becomes comparable to the MF (no-fluctuation) prediction. The field h_c^{fluct} which limits the fluctuation-dominated region at $T = T_C$ has to be contrasted with the field h_c^{MF} considered in the present paper, which limits the scaling regime of the MF theory, i.e. the regime where the MF solution for $m(\tau,h)$ may be well represented by Eq. (1). The quantity h_c^{fluct} was estimated in Ref. [14] from the series expansion for the field dependence of the magnetization on the critical isotherm, from the spherical model, and from the Gaussian model. For ordinary ferromagnets such as Fe, Ni and Co, the Gaussian model yields the estimate $h_c^{\text{fluct}} \approx 10^{-3}$, i.e. h_c^{fluct} is of the same order of magnitude as h_c^{MF} from Eq. (33). This coincidence for the above ferromagnets means that there is no domain of values of h over which MF scaling in applied field can be observed. In other words, h_c^{MF} and h_c^{fluct} are two characteristic fields, the latter of which has to be estimated in each particular case. MF scaling behavior in the presence of external field could be expected to be observable only in the case when $h_c^{fluct} < h_c^{MF}$; in fact, in view of the experimental difficulties in probing the asymptotic critical region, it should be required that a 'much less' inequality should hold. There are certainly cases where this inequality is expected to hold. Apart from $HoRh_4B_4$, as mentioned above [12], very recent careful measurements carried out in the asymptotic critical region of $ZrZn_2$ have led to the unambiguous estimate that the critical exponents γ, β and δ in this system have the MF values of 1, 1/2 and 3, to within an experimental error of 5% [21]. It would be of great value if measurements and analysis of the maximum of the field-dependent susceptibility were carried out for these two 'mean-field' systems; in particular, this would contribute to the elaboration of the advantageous method of determining the critical exponents via such measurements [3–5,7,8].

A detailed description of the possible crossovers between different regimes must necessarily resort to the introduction and eventual computation of *effective* critical exponents in crystalline and amorphous ferromagnets [21–25]. However, this lies beyond the scope and purpose of the present paper.

4.3. The susceptibility MF scaling function for any S

Let us emphasize once again that the parametric method allows one to study the MF field-dependent susceptibility for *any* values of spin, temperature, and field and not only in the scaling region. Moreover, the parametric method enabled us to estimate the limiting value of applied field above which deviations from scaling laws set in. For example, it is just as easy to obtain the susceptibility as a function of temperature in *fixed* field and for *different* values of S. This is illustrated in Fig. 2, where we give the normalized susceptibility for $h = 10^{-3}$ and different S. Bearing in mind the proximity of T_m and T_C for small fields, the sensitivity of the location of t_m to

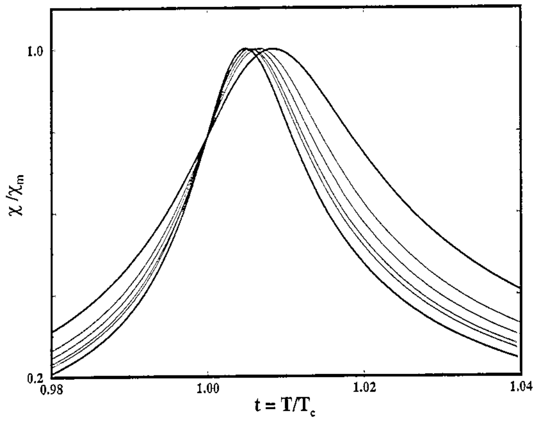

Fig. 2. Normalized susceptibility above and below T_C in fixed field $h = 10^{-3}$. Thick lines are limiting cases of $S = \frac{1}{2}$ (whose maximum is to the right) and $S = \infty$. In between: $S = 1$, $\frac{3}{2}$, $\frac{5}{2}$ and $\frac{7}{2}$.

changes in the spin value is astonishingly large. It should also be mentioned in passing that the analysis of the classical case of $S = \infty$ proceeds quite analogously to what has been given so far for finite values of S. One must simply substitute the Langevin function $L(x) = \coth(x) - 1/x$ and its derivatives for the Brillouin function $B_S(x)$ and its derivatives; besides, the limiting ($S \to \infty$) values of the spin-dependent constants $a(S) \to 1$, $b(S) \to 0$, and $c(S) \to 3$ have to be inserted wherever appropriate.

The normalized MF scaling function for the field-dependent susceptibility can now be obtained by normalizing $\chi(\tau,h)$ against $\chi(\tau_m,h)$ as prescribed by Eq. (11). At T_C and for $h \neq 0$, one finds

$$\bar{F}(0) = \frac{\chi(0,h)}{\chi(\tau_m,h)} = \frac{1}{\sqrt[3]{2}} = 0.7937. \qquad (34)$$

Note that $\bar{F}(0)$ is a universal, *spin-independent* constant. The implementation of the parametric approach to compute $\bar{F}(\sigma)$ according to Eq. (11) is straightforward. However, one must be careful to use the information obtained in Sections 4.1 and 4.2 concerning the limitations to MF scaling, imposed by the applied field. By criterion (33), values of h smaller than 10^{-3} have to be chosen in the parametric sweep. Otherwise, non-universal behavior for $\bar{F}(\sigma)$ would show up, signaling that one is no longer in the asymptotic MF regime. In practice, the compu-

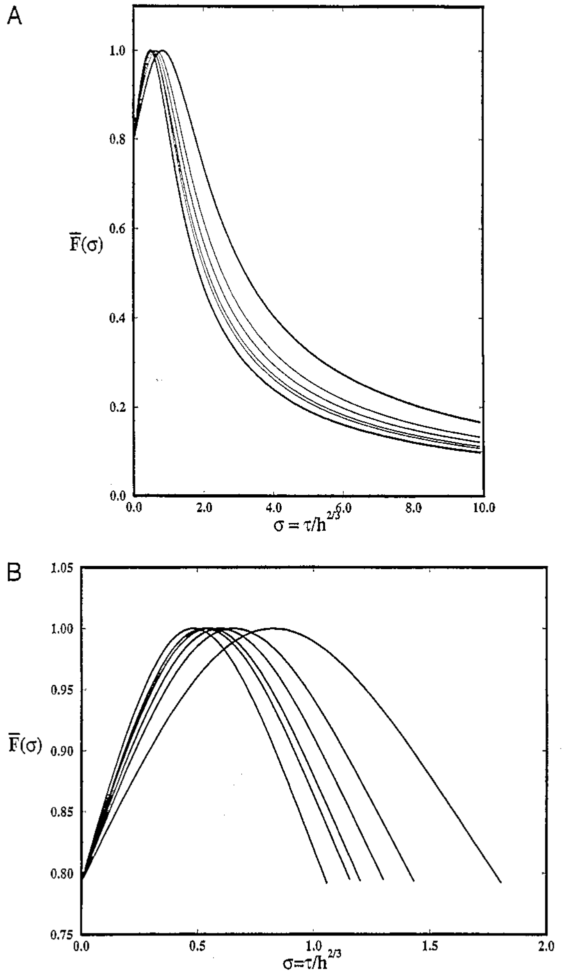

Fig. 3. (a) Critical scaling functions $\overline{F}(\tau/h^{2/3})$ for the MF susceptibility in applied field as found by means of the parametric approach. Thick lines: $S=\frac{1}{2}$ (right) and $S=\infty$ (left); in between: $S=1$, $\frac{3}{2}$, $\frac{5}{2}$ and $\frac{7}{2}$. (b) Enlargement of the scaling function $\overline{F}(\sigma)$ in the vicinity of its maximum. Curves arranged as in (a).

tation of the scaling function in different sweeps with $h=10^{-3}$, 10^{-4} and 10^{-5}, respectively, gives *identical* curves, for a given value of S, for the spin-specific scaling function $\overline{F}(\sigma)$ for values of σ which encompass the most interesting domain with the maximum of the susceptibility.

The MF scaling function $\overline{F}(\tau/h^{2/3})$ for different values of spin S as found by the parametric method and including the limiting values of $S=\frac{1}{2}$ and $S=\infty$, is presented for the first time in Fig. 3(a,b). All curves coalesce at the value of $\overline{F}(0)=1/\sqrt[3]{2}$ at $T=T_C$, i.e. at $\sigma=\tau=0$. The location of the maximum in Fig. 3(a,b) is found to be

$$\sigma_m=\left(\frac{9}{80}\right)^{1/3}(a^2+b^2)^{1/3}$$

$$=\left(\frac{9}{80}\right)^{1/3}\left(\frac{2S^2+2S+1}{2S^2}\right)^{1/3}. \quad (35)$$

This decreases monotonically with increasing S from $5^{1/3}\cdot(9/80)^{1/3}$ at $S=\frac{1}{2}$ to $(9/80)^{1/3}$ at $S=\infty$. The shift of σ_m from the classical limit to the extreme quantum limit is thus by a factor of $5^{1/3}$.

The form of the scaling function $\overline{F}(\sigma)$ in the vicinity of its maximum suggests that a quadratic approximation may be quite successful in this region. To test this possibility, we construct the quadratic approximant $\overline{\mathscr{F}}(\sigma)=c_0+c_1\cdot\sigma+c_2\cdot\sigma^2$ by requiring that it match $\overline{F}(\sigma)$ at zero and σ_m, and that the latter point be the location of the maximum of $\overline{\mathscr{F}}(\sigma)$ as well. These three conditions provide for three independent algebraic equations for the coefficients of the approximant and lead to

$$\overline{\mathscr{F}}(\sigma)=c_2(\sigma-\sigma_m)^2+1, \quad (36)$$

with

$$c_2=\left(\overline{F}(0)-1\right)/\sigma_m^2. \quad (37)$$

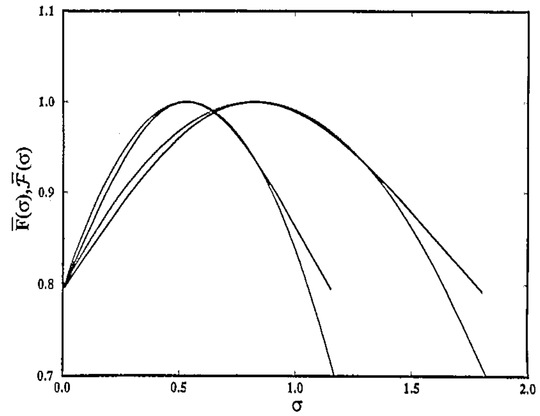

Fig. 4. Nonlocal parabolic approximants for $S=\frac{1}{2}$ (maximum to the right) and $S=\frac{7}{2}$ to the critical scaling function (cf. Eqs. 36, 37). Thick lines: exact scaling function $\overline{F}(\sigma)$; dotted lines: approximant $\overline{\mathscr{F}}(\sigma)$.

Note that the approximation is nonlocal in that it is pinpointed at two distinct values of the argument for which the scaling function $\bar{F}(\sigma)$ is exactly known. The approximation is only reasonably successful and is illustrated in Fig. 4 for $S=\frac{1}{2}$ and $\frac{7}{2}$. Obviously, local quadratic matches for the vicinities of $\sigma=0$ and $\sigma=\sigma_m$ may be better, but would surely be satisfactory only sufficiently close to the respective points. This is a corollary of the fact, emphasized above, that $\sigma_m \sim O(1)$; hence, a local expansion around $\sigma=0$ with only a few terms kept would widely miss the maximum, while a similar expansion about $\sigma=\sigma_m$ would widely miss the correct value of the scaling function at $\sigma=0$. This point is further illuminated in Section 5. A further possibility is to construct higher-order approximants to match the scaling function globally. In fact, we now proceed to propose an analytic insight which, among other things, allows one to control such higher-order approximants, should they be required.

5. A complementary approach to the explicit determination of the MF scaling functions for the magnetization and the susceptibility

A decisive advantage of the parametric approach reported above is that it is effective with *any* values of the thermodynamic parameters h and τ. The scaling function for the susceptibility was found as a particular and, in fact, asymptotic case for $h \le h_c^{MF} = 10^{-3}$ and the determination of h_c^{MF} was based on observations of the extent of validity of MF scaling for quantities whose field dependence was generated within the parametric approach.

The problem now is whether some sort of analytic handling of the MF scaling region can be suggested that would go so far as to produce reliable information about the maximum of the susceptibility which lies in a region, inaccessible with sufficient accuracy to few-term expansions of the scaling function. We have found a very simple possibility that seems to have escaped attention.

One starts with the MF equation for the magnetization (13). Once again, keeping track of the orders of magnitude of the variables involved is paramount to the correct description of the maximum of the susceptibility. For sufficiently small fields h and above T_C, the generalized effective field $x=(cm+h)/(1+\tau)$ is small, too. Expanding the Brillouin function to the first non-trivial order gives

$$m = A\cdot x - B\cdot x^3 + O(x^5), \tag{38}$$

where

$$A = 1/c = \frac{a+b}{3}, \qquad B = \frac{(a+b)(a^2+b^2)}{45}. \tag{39}$$

In order to proceed further, one is forced to make an assumption about the order of magnitude of the temperature variable τ in terms of the applied magnetic field. Otherwise, one is not in the position to know whether an expansion of the denominator $1+\tau$ in the generalized effective field is allowed and, if so, how many terms of it have to be kept. Indeed, the necessity of binding the two otherwise independent thermodynamic variables τ and h anticipates the relevance of only their scaling combinations or σ in the scaling region; this combination is trivially the ratio h/τ in the paramagnetic case only [11]. Assuming that τ is small and recalling that $m \sim h^{1/3}$ on the critical isotherm, one sees that a systematic expansion of x and, hence, of the whole problem, requires that τ be of the order of $h^{2/3}$. Under these explicitly formulated assumptions, one finds to the lowest non-trivial order, and this is $O(h^1)$, that

$$Bc^3m^3 + \tau m - Ah = 0. \tag{40}$$

Now divide both sides of the equation by $(Bc^3)\cdot h$, define $\overline{m} \equiv m/h^{1/3}$ and recall that $\sigma = \tau/h^{2/3}$, so that

$$\overline{m}^3 + \frac{1}{Bc^3}\sigma\cdot\overline{m} - \frac{A}{Bc^3} = 0. \tag{41}$$

This equation for $\overline{m}$ has a unique positive real root by virtue of Descartes' criterion [26]. Moreover, since the root of this cubic depends on its coefficients only with A, B, c and σ being spin-dependent constants, one finds immediately that

$$\overline{m} = f(\sigma;S), \text{ hence } m = h^{1/3}\cdot f(\sigma;S), \tag{42}$$

where, of course, the root $f(\sigma;S)$ is just the scaling function for the magnetization as defined in Eq. (2). We have thus found an explicit expression for the scaling function for the magnetization. Defining $p \equiv \sigma/Bc^3$ and $q \equiv -A/Bc^3$, we come to the reduced form of the cubic $\overline{m}^3 + p \cdot \overline{m} + q = 0$, hence, the scaling function is given explicitly by

$$f(\sigma;S) = \left(-q/2 + Q^{1/2}\right)^{1/3} + \left(-q/2 - Q^{1/2}\right)^{1/3}, \tag{43}$$

with

$$Q = (p/3)^3 + (q/2)^2. \tag{44}$$

If one is interested in the scaling function $g(s)$ in terms of the scaling variable s, one can use the connection between s and σ or, even more easily, one can straightforwardly manipulate Eq. (40) in the same fashion as above, this time dividing both sides by $\tau^{1/2}$ and introducing naturally the variable $s = h/\tau^{3/2}$.

As particular cases one can find the magnetization on the critical isotherm and, indeed, its related critical exponent $\delta = \frac{1}{3}$ by setting $\sigma = 0$ in the scaling function $f(\sigma)$; just as easily, one deduces the MF critical behavior and the exponent $\beta = \frac{1}{2}$ in zero field by setting $s = 0$ in the explicit scaling function $g(s)$. The scaling functions for the susceptibility in the form $F(\sigma)$ or $G(s)$ are trivially obtained by differentiating the corresponding explicit functions for the magnetization with respect to the field. As a simple corollary, one obtains, from the form with the function $G(s)$, the critical MF singularity with exponent γ of the susceptibility in zero field. Finally, the maximum of susceptibility can be studied quite easily by using the scaling form with the function $F(\sigma)$. The asymptotic scaling functions obtained in both described approaches are the same, i.e. the plots generated in both ways literally coincide.

Now that we have the explicit analytic expression (43) for the asymptotic regime, we return to the point of studying the error incurred by attempting to describe the maximum of the susceptibility by keeping only a few terms in an expansion of the scaling function. The effect is illustrated in Fig. 5, where one can see the plots for the successive approximations to the asymptotic scaling functions for the susceptibility for spin $S = \frac{1}{2}$ (series approximations for the function $\overline{F}(\sigma)$ from Eq. (11). As the expansions have been taken about the point $\sigma = 0$, we have series of local approximations about the said point. It is hardly necessary to comment further on the substantial deviations that occur even with terms of $O(\sigma^5)$ taken into account. The local approximations for $\overline{F}(\sigma)$ can be compared with the nonlocal approximant proposed in the previous section (see also Fig. 4). The expectation that the expansions would also fail to detect to a satisfactory accuracy the location of the maximum is also fulfilled: in the example of $S = \frac{1}{2}$, the exact value of σ_m as given by Eq. (35) is 0.8255, while working by expanding Eq. (6) for $\sigma \to 0$ gives 0.9302, 0.7242 or 0.7898 when keeping one, two, or three σ-dependent terms in the expansion, respectively.

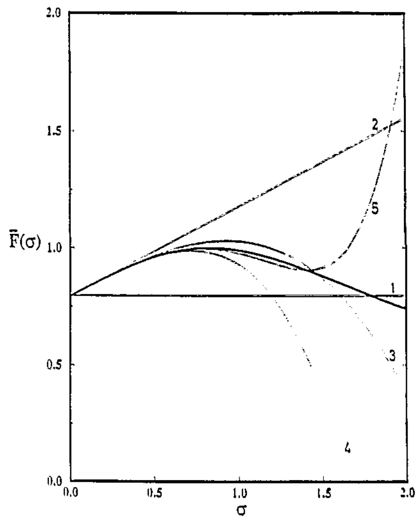

Fig. 5. Insufficiency of few-term expansions about $\sigma = 0$ to describe the scaling function in the region $\sigma \sim O(1)$ where the maximum of susceptibility occurs. Successive series expansions for the susceptibility scaling function $\overline{F}(\sigma)$: the thick curve is the exact function for $S = \frac{1}{2}$. The enumeration of curves corresponds to the inclusion of successive non-vanishing terms in the expansion.

6. Summary

We have presented an exhaustive analysis of the field-dependent susceptibility of ferromagnets in mean-field theory. Special attention has been dedicated to the description of the scaling with field of the susceptibility maximum occurring above T_C and of its location on the temperature scale. The considerable interest to analyze the MF predictions is twofold: (i) theoretically, only the trivial and academic case of the field-dependent susceptibility of a paramagnet had been described so far; (ii) experimentally, a method based on the general scaling theory predictions which allows us to measure two critical exponents independently has been successfully implemented [3–8,19] with the very serious advantage over alternative methods that, with it, the critical temperature T_C does not need to be known very precisely.

Two different novel methods of analysis of the field-dependent quantities in the MF theory are presented. The *parametric* method is related to similar ones recently proposed to study different aspects of ferromagnetism, antiferromagnetism, and magnetic anisotropy [16–18]. It provides an unbeatable and easy-to-implement tool for the study of all aspects of MF field-dependent magnetization and susceptibility for any spin value S and for any field and temperature. The asymptotic scaling functions are obtained within this method as particular cases for sufficiently small fields, $h \leq 10^{-3}$. The *analytic* method addresses the description of the MF asymptotic region only and is seen to result from a very simple manipulation of the self-consistent equation for the magnetization which seems to have escaped attention, even though it provides for a unified description of the scaling function involved.

The central results of the paper are grouped around the determination of the scaling functions $f(\sigma)$ for the magnetization and $\bar{F}(\sigma)$ for the susceptibility. They are explicitly given and analyzed. Special emphasis was laid on the description of the most pronounced experimental feature, the maximum of susceptibility. We have shown that the fact that it occurs at a value of the scaling variable σ_m which is of the order of unity, regardless of the spin value, has the consequence that the maximum and its location cannot be adequately described by means of series expansions of the scaling functions about $\sigma = 0$. Among other things, we have proposed a nonlocal parabolic approximant to the susceptibility scaling function. While it is not a match or substitute for the parametric solution, for instance, its simplicity may offer expedience in analytic estimates with the exact location and height of the susceptibility maximum.

It is further suggested that experimentally known cases of 'mean-field' ferromagnets such as $HoRh_4B_4$ [12] and $ZrZn_2$ [21] are studied with the purpose of determining the scaling functions experimentally. Furthermore, the methods of this paper could tentatively be used in the study of other ferromagnetic materials in which long-range forces are suspected to dominate and, hence, to lead to MF scaling behavior.

Apart from this, the parametric method has enabled us to estimate the width of the MF scaling region in an applied field, thus providing a reference case for comparison of the results of theories accounting for the fluctuations of the order parameter (magnetization) in the critical region.

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