Magnon nodal-line semimetals and drumhead surface states in anisotropic pyrochlore ferromagnets

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(Received 30 September 2016; revised manuscript received 16 December 2016; published 17 January 2017)

DOI: 10.1103/PhysRevB.95.014418

I. INTRODUCTION

Recently, nontrivial topologies of magnon spectra have become a thriving field of research. In striking analogy to electronic topological matter [1], topological magnon matter has been identified. The drosophilae of such topological magnon insulators (TMIs) [2], which are the pendant to electronic Chern insulators, are (two-dimensional) ferromagnets on a kagome lattice with Dzyaloshinskii-Moriya interaction (DMI) [3–10]. This last causes complex hopping matrix elements in the free-boson Hamiltonian of magnons and thus breaks the time-reversal symmetry; this points towards the textured magnetic flux in the Haldane model [11]. As a result, Berry curvatures and Chern numbers are nonzero and cause topologically protected edge magnons. The last of these revolve unidirectionally the sample in accordance with the bulk-boundary correspondence [12,13]. Recently, Cu-(1,3-benzenedicarboxylate) was identified as a TMI which well [15].

The quest for topologically nontrivial systems has been initiated by the discovery of the magnon Hall effect [16,17] in ferromagnetic pyrochlore oxides, mostly because the transverse thermal Hall conductivity has been related to the Berry curvature of the bulk magnons [5,18,19]. The quite natural extension of the topological classification to three-dimensional systems led to the discovery of magnon Weyl semimetals [20,21], in which the crossing points of two magnon bands act as source and sink of Berry flux (again in close analogy to electronic systems [22,23]).

Here we complete the family of topological magnonic objects in three-dimensional ferromagnetic materials by predicting magnon nodal-line semimetals (magnon NLSMs), the magnon pendant of electronic NLSMs [24–30]. For this purpose, we consider a ferromagnetic pyrochlore lattice with anisotropic exchange interactions, but without spin-orbit interaction (SOI). We find two nodal lines, that is, two closed loops in reciprocal space along which two magnon bands are degenerate. On top of this, we identify the protecting symmetries and calculate the topological invariants of the nodal lines. Magnon spectra for the (111) surface feature drumhead surface states, i.e., the hallmarks of NLSMs. The dispersion relations of these depend strongly on the termination of the surface. Eventually, we discuss the effect of a nonzero DMI on the spectra and suggest experiments to identify magnon NLSMs.

II. MODEL AND SPIN-WAVE ANALYSIS

The pyrochlore lattice consists of four interpenetrating face-centered cubic (fcc) lattices, resulting in a regular array of corner-sharing tetrahedra [see Fig. 1(a)]. Along the [111] direction, kagome layers alternate with triangular layers.

Considering only nearest-neighbor Heisenberg exchange interactions, there is only coupling between sites within a kagome layer [$J_N$, solid bonds in Fig. 1(a)] and between sites in a triangular layer with sites in the adjacent kagome layers [$J'_N$, dashed bonds in Fig. 1(a)]. To introduce anisotropic exchange, we assume that $0 < J'_N < J_N$. Thus, the Hamiltonian

$$H = -J_N \sum_{\langle ij \rangle} \mathbf{s}_i \cdot \mathbf{s}_j - J'_N \sum_{\langle ij \rangle} \mathbf{s}_i \cdot \mathbf{s}_j$$

includes only a symmetric exchange between spins $s_i$ and $s_j$ at sites $i$ and $j$, respectively. An external magnetic field is not considered because it merely shifts the magnon spectrum towards higher energies. Moriya’s symmetry rules [4] would, in principle, allow for a nonzero DMI [31,32] but for the time being we consider vanishing SOI.

A truncated Holstein-Primakoff transformation [33]

$$s_i^+ \rightarrow s - n_i,$$  

$$s_i^+ = s_i^z + i s_i^x \rightarrow \sqrt{2} s_i^z a_i,$$  

$$s_i^- = s_i^z - i s_i^x \rightarrow \sqrt{2} s_i^z a_i^\dagger,$$

is applied (linear spin-wave approximation, i.e., no magnon-magnon interactions). The annihilation operators $a_i$ and the creation operators $a_i^\dagger$ obey the boson commutation rules; $[n_i, a_j] = 0$, $[a_i, a_j^\dagger] = 0$, $[a_j, a_j^\dagger] = 1$, $[n_i, a_j^\dagger] = [n_i, a_j] = 0$, $[n_i, n_j] = 0$.
site 3 connects adjacent kagome layers by a weaker interaction. A kagome lattice with nearest-neighbor interaction $J$ approximation yields the free-magnon Hamiltonian matrix $H_{\text{free}}$ of the anisotropic pyrochlore ferromagnet along the red path depicted in (b) for $J_S = 1$ meV, $\lambda = 0.8$, and $s = 1/2$. Band crossings are marked $N_{12}$ and $N_{23}$, with insets showing magnifications. The central inset displays the spectrum for the isotropic case ($\lambda = 1$).

The magnon band structure is made up of four bands, consistent with the four basis atoms of the pyrochlore lattice [Fig. 1(b)] shows the high-symmetry lines along which the spectrum is shown in Fig. 1(c)]. For isotropic exchange ($\lambda = 1$; see central inset), the two topmost bands are dispersionless and the lower two bands are mirror images of each other. For $\lambda < 1$ ($\lambda = 0.8$ in Fig. 1(c)], one of the formerly dispersionless bands (green) becomes dispersive; this leads to a crossing with band 2 (red; counted from below) at a single $k$ point on the $\Gamma$–K line (marked $N_{23}$). Additionally, the crossing $N_{12}$ of band 1 (blue) and band 2 (red), which is at the W point for $\lambda = 1$, is shifted along the W–L line for $\lambda < 1$. These degeneracies are part of the nodal lines shown in Fig. 2. In the following we label the nodal lines by $N_{\mu\nu}$, in which $\mu$ ($\nu$) stands for the energetically lower (higher) band forming the nodal line.

$N_{12}$ is best depicted on the cube circumscribing the fcc Brillouin zone [BZ; Fig. 2(a)]. It does not lie exactly on the cube’s faces but is set off by a tiny amount towards the BZ center. For decreasing $\lambda$, $N_{12}$ keeps contracting until it becomes more and more ring-like; it is moved towards the hexagonal faces of the BZ until it vanishes in the L point for $\lambda = 0$ (which is the limit of noninteracting kagome layers).

$N_{23}$ has its center at the $\Gamma$ point [Fig. 2(b)], its diameter increases for decreasing $\lambda$. It too is not a planar ring in $k$ space, but shows modulations that are consistent with the threefold rotational symmetry of the [111] direction ($\Gamma$–L line).

Please note that the third and fourth bands are degenerate along $\Gamma$–L [green and brown bands in Fig. 1(c)], forming an “open” line of degeneracies along a high-symmetry direction (“open” means in the same sense as open orbits on a Fermi surface). This line is, however, not topologically protected (see the following section); its shape is not dependent on $\lambda$.

III. SYMMETRY AND TOPOLOGY ANALYSIS

Nodal lines are differentiated by their protecting symmetry $\mathcal{S}_{\text{top}}$. A type-1 nodal line is protected by mirror symmetry and thus has to lie within a mirror plane. A type-2 nodal line is protected by the simultaneous presence of inversion and time-reversal symmetry in a system without SOI; it may appear in generic positions in $k$ space. In the following we show that the magnon nodal lines are of type 2.

The free-magnon Hamiltonian considered here is time-reversal invariant because there is no complex hopping (in contrast to the case with DMI $\mathcal{S}_{\text{top}}$); due to the ferromagnetic ground state, inversion symmetry is also present. In combination, these symmetries imply that the Berry curvature $\mathcal{S}_{\text{top}}$

$$\Omega_{\alpha}(k) = i\langle \nabla_k u_{\alpha}(k) | \times | \nabla_k u_{\alpha}(k) \rangle$$

(5)
and consequently the Chern numbers

\[ C_n = \frac{1}{2\pi} \oint_S \Omega_n(k) \cdot e \, dS \]  

vanish, thereby suggesting a topologically trivial situation (\( \epsilon \) is the \( k \)-dependent local normal of a closed surface \( S \)). However, the nontrivial topology of a nodal line is identified by the Berry phase integrated along an arbitrary closed loop \( C \) [29,30],

\[ \gamma_n[C] = i \int_C \langle u_n(k) | \nabla u_n(k) \rangle \, dk. \]  

If \( C \) and the nodal line are intertwined (sketched in Fig. 3), \( \gamma_n[C] = \pi \) (nontrivial), otherwise \( \gamma_n[C] = 0 \) (trivial).

To prove the nontrivial topology of the nodal lines, we compute \( \gamma_n[C(r)] \) of the first and third bands in dependence on the radius \( r \) of a circle \( C \). Figure 3 shows as an example \( \gamma_\epsilon[C(r)] \), but we note that the argument is valid also for \( \gamma_\epsilon[C(r)] \). The center of \( C \) is chosen such that it is well separated from the nodal line \( N_{23} \). Hence, the nodal line does not puncture the surface enclosed by \( C \) and \( \gamma_\epsilon[C(r)] = 0 \) (left inset in Fig. 3). The increasing of \( r \) leads to the interweaving of the nodal line and \( C \) (right inset) once the critical radius \( r_o \) is reached: \( \gamma_\epsilon[C(r)] = \pi \). Upon a further increase of \( r \) the nodal line and \( C \) become separated again and \( \gamma_\epsilon[C(r)] \) falls back to zero for \( r > r_o \). This topology analysis reveals that the nodal lines are of type 2. Please see the Appendix for a detailed discussion of the \( \mathbb{Z}_2 \) topological invariants associated with the nodal lines.

The degeneracy of bands 3 and 4 along \( \Gamma-L \) is accidental: the Berry phase \( \gamma_\epsilon \) of the fourth band is zero for any loop \( C \). An arbitrarily small perturbation which preserves symmetry would lift the degeneracy [30]; e. g., one could vary the exchange interaction between any of the kagome sites 1, 2, or 4 [see Fig. 1(a)].

IV. SURFACE STATES

The surface magnon dispersion is analyzed in terms of the spectral density \( N_p(\epsilon,k) \) which is computed for a semi-infinite geometry by Green’s function renormalization [37]. The renormalization proceeds as follows. For the chosen (111) surface, the pyrochlore lattice is decomposed into principal layers (PLs) which are parallel to that surface. The principal layers are chosen in such a way that the Hamiltonian matrix of the semi-infinite system comprises only interactions within a PL and among adjacent PLs. In the infinite set of equations for the PL-resolved Green’s-function matrix the inter-PL interactions are iteratively reduced (renormalized). After a few iterations the entire Hamiltonian matrix becomes effectively block diagonal which allows to compute the spectral densities

\[ N_p(\epsilon,k) = -\frac{1}{\pi} \lim_{\hbar \to 0^+} \text{Im}[\text{Tr} G_{pp}(\epsilon + i\hbar,k)] \]  

of PL \( p \) from the Green’s function block \( G_{pp} \). A finite \( \eta \) (here 0.001 meV) ensures convergence and introduces broadening. Hence we have access to the bulk spectral density (\( p = \infty \)) to that of any other PL, in particular that of the surface (\( p = 0 \)).

The perspective views given in Fig. 2 in which \( \Gamma \) and \( L \) coincide [right in Fig. 2(a); left in Fig. 2(b)] reflect the rotational symmetry of both the nodal lines and the hexagonal BZ of the (111) surface. The argumentation given below is also valid for other surfaces with nontrivial projections of the nodal lines.

The bulk magnon spectrum appears as broad features upon projection onto the (111) surface (cf. the extended green regions in Fig. 4); surface states show up as comparatively sharp features. Constant-energy cuts of the surface magnon spectrum at energies close to the nodal line \( N_{23} \) are plotted in Fig. 4(a) [the corresponding energies are indicated in Fig. 4(b)]. At the energy for which only \( N_{23} \) is present, the projected bulk states form a closed ring in the surface BZ [top constant-energy cut in Fig. 4(a)]. Additionally, \( DSS_{23} \) which is associated with \( N_{23} \) produces a ring-like feature whose extension shrinks toward \( \mathbb{K} \) with decreasing energy. The cut of the surface magnon spectrum along high-symmetry lines of the surface BZ [Fig. 4(b)] shows two points of \( N_{23} \): one on the \( \Gamma-M \) and another on the \( \Gamma-K \) line. \( DSS_{23} \) is suspended at these points, its considerable dispersion indicates that the ‘membrane is stretched quite loosely’. Figures 4(c) and 4(d) show the same scenario for the projection of \( N_{12} \) and \( DSS_{12} \) with decreasing energy, \( DSS_{12} \) shrinks towards \( \Gamma \).

The (111) surface of the pyrochlore lattice allows for two terminations: a kagome or a triangular layer. The results discussed up to here were obtained for the triangular termination. Considering the kagome termination [Figs. 4(e) and 4(f)], there is no difference in the bulk contributions but a major variation in the \( DSS \), here exemplified by \( DSS_{12} \). If we call the region about the \( \Gamma \) point the inside of the projected \( N_{12} \), we find \( DSS_{12} \) now in its exterior. This fundamental dependence on the surface termination has also been observed for electronic NLSMs, for example, in the alkaline-earth stannides, germanides, and silicides [29].

V. DISCUSSION

A. Effect of the Dzyaloshinskii-Moriya interaction

For the electronic NLSMs, SOI has to be absent [29,34] or at least very weak because otherwise the nodal lines would be lifted and other topological states could occur. The same is valid for the magnon nodal lines: the pyrochlore lattice
nonzero magnetic field rather than by DMI \cite{39–41}. The free-magnon Hamiltonian by a nontrivial spin chirality in frustrated antiferromagnets, a (synthetic) SOI is introduced to lines could not exist \cite{49}. Instead, the magnon bands would and time-reversal symmetry would be removed and nodal points \cite{21,38}. Other topological states are likely, e.g., Weyl carry nonzero Chern numbers and topological surface states.

FIG. 4. Magnons at the (111) surface of an anisotropic pyrochlore ferromagnet for two surface terminations; Panels (a–d) show results for a triangular termination (top and central row), while (e) and (f) show results for a kagome termination (bottom row). The surface spectral density $N_0(\varepsilon, \mathbf{k})$ is represented as a color scale (black: zero; white: maximum). Bulk magnons appear as broad features, surface states as sharp light lines. Panels (a), (c), and (e) show constant-energy cuts through the entire surface Brillouin zone for energies indicated by magenta lines in (b), (d), and (f), respectively. Panels (b), (d), and (f) display spectral densities along high-symmetry directions of the surface Brillouin zone. The projected nodal lines $N_{12}$ and $N_{23}$ and the respective drumhead surface states DSS$_{12}$ and DSS$_{23}$ are marked. Parameters as in Fig. 1(c).

allows for a nonzero DMI which would break the time-reversal symmetry \cite{21}. Thus, the inevitable combination of inversion and time-reversal symmetry would be removed and nodal lines could not exist \cite{49}. Instead, the magnon bands would carry nonzero Chern numbers and topological surface states would appear. Other topological states are likely, e.g., Weyl points \cite{21,38}.

The above discussion applies solely to ferromagnets. For frustrated antiferromagnets, a (synthetic) SOI is introduced to the free-magnon Hamiltonian by a nontrivial spin chirality in nonzero magnetic field rather than by DMI \cite{39–41}.

B. Experimental considerations

To prove the existence of magnon nodal lines, one could utilize either a direct or an indirect approach.

Considering a direct mapping, e.g., by inelastic neutron scattering for the bulk magnons \cite{42} and electron energy loss spectroscopy for the surface magnons \cite{43}, we propose to tune the anisotropy (imbalance of $J_N$ and $J'_N$) by applying strain. As the diameter of the nodal line is sensitive to the ratio $J'_N/J_N$, it will increase or decrease accordingly; such a behavior can be detected in the experiments. If samples with different surface termination can be grown, one could utilize the strong dependence of the DSSs on the termination [cf. Figs. 4(d) and 4(f)].

In view of an indirect approach, one could think of transverse transport, for instance, the magnon Hall effect. However, these effects require a nonzero Berry curvature which is ruled out by the simultaneous presence of inversion and time-reversal symmetry required for an NLSM. We recall that most of the ferromagnetic pyrochlores exhibit a sizable DMI; examples are Lu$_2$V$_2$O$_7$, In$_2$Mn$_2$O$_7$, and Ho$_2$V$_2$O$_7$ \cite{16,17}. Thus, the measurement of the magnon Hall effect can be used for the identification of samples with negligible DMI, which are likely to exhibit magnon nodal lines; possible candidates include the ferromagnetic pyrochlore manganites \cite{44} and the chromium spinels \cite{45}.

The Hamiltonian (1) is surprisingly simple as it involves only an exchange interaction, which suggests that magnon NLSMs could be quite common and not restricted to the pyrochlore lattice. The examination of this conjecture is beyond the scope of this study which serves as a proof of principle; further theoretical and experimental work is clearly necessary. However, one rule can be readily formulated: the magnetic unit cell must contain more than one atom, which excludes Bravais lattices.

VI. CONCLUSION

With the prediction of magnon NLSMs the correspondence of electronic and magnonic topologically nontrivial systems in ferromagnets is completed. The term “semimetal” does not respect the bosonic nature of magnons and therefore is, strictly speaking, meaningless. However, the topological features—here: nodal lines and drumhead surface states—exist irrespective of fermion or boson statistics. The pendant of topological Dirac semimetals cannot be realized for magnons in ferromagnets because the twofold degeneracy of the bands is forbidden.

ACKNOWLEDGMENT

This work is supported by SPP 1666 of Deutsche Forschungsgemeinschaft (DFG).

APPENDIX: $\mathbb{Z}_2$ TOPOLOGICAL INVARIANTS FOR NODAL LINES

Another way to identify nodal lines in inversion-symmetric crystals has been presented in Ref. \cite{28}: Kim et al. introduced $\mathbb{Z}_2$ invariants based on parity eigenvalues at parity-invariant momenta (or time-reversal invariant momenta, TRIMs)

$$
\Gamma_{i=n_1n_2n_3} = \frac{1}{2}(n_1b_1 + n_2b_2 + n_3b_3) \tag{A1}
$$

$(n_j = 0, 1), b_j$ reciprocal lattice vector, and $j = 1, 2, 3$). The Berry phase $\omega$ on a contour $C_{ab}$ which connects two TRIMs $\Gamma_a$ and $\Gamma_b$ and which is built from two time-reversed paths is
TABLE I. Parity eigenvalues $\xi_i$ (quarter filling) and $\xi_i\xi_j$ (half filling) at the eight time-reversal invariant momenta (TRIMs) $\Gamma_i$, with $i = \{n_1,n_2,n_3\}$; cf. Eq. (A1).

<table>
<thead>
<tr>
<th>TRIM</th>
<th>$n_1$</th>
<th>$n_2$</th>
<th>$n_3$</th>
<th>$\xi_1$</th>
<th>$\xi_1\xi_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\Gamma_1$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>+</td>
<td>+</td>
</tr>
<tr>
<td>$\Gamma_2$</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>+</td>
<td>-</td>
</tr>
<tr>
<td>$\Gamma_3$</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>+</td>
<td>-</td>
</tr>
<tr>
<td>$\Gamma_4$</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>+</td>
<td>-</td>
</tr>
<tr>
<td>$\Gamma_5$</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>$\Gamma_6$</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>+</td>
<td>-</td>
</tr>
<tr>
<td>$\Gamma_7$</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>+</td>
<td>-</td>
</tr>
<tr>
<td>$\Gamma_8$</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>+</td>
<td>-</td>
</tr>
</tbody>
</table>

The parity eigenvalues $\xi_i$ and $\xi_i\xi_j$ are given by

$$\omega(C_{ab}) = \exp \left( \frac{1}{n} \sum_{\nu} \gamma_{\nu}[C_{ab}] \right) = \xi_a\xi_b,$$

(A2)

in which $\xi_n = \prod_{\nu} \xi_n(\Gamma_\nu)$ and $\xi_n(\Gamma_\nu)$ is the parity eigenvalue of the nth band at $\Gamma_\nu$. For electrons, one would restrict the sum and the product over $n$ to the occupied states; for magnons (or generally for bosons), the product is over all bands below the first and second (second and third) bands, the product is over the first (first and second) band. For convenience, we adopt the fermionic terms “quarting filling” and “half filling” although they have no meaning for magnons.

Following Ref. [28], the Berry phase of a contour connecting four TRIMs reads $\omega(\partial S_{abcd}) = \xi_a\xi_b\xi_c\xi_d$, which counts the number $N(S_{abcd})$ of nodal lines that pierce the enclosed surface $S_{abcd}$ from

$$(-1)^{N(S_{abcd})} = \omega(\partial S_{abcd}),$$

(A3)

follows that $\xi_a\xi_b\xi_c\xi_d = -1$ indicates an odd number of nodal lines piercing through $S_{abcd}$.

The parity eigenvalues are evaluated as proposed by the authors or Ref. [46]: choosing site 3 as a center of inversion [see Fig. 1(a)], the parity operator in momentum space reads

$$P_k = \text{diag}(e^{-ia_1\cdot k}, e^{-ia_2\cdot k}, 1, e^{-ia_3\cdot k}),$$

(A4)

with $a_1 = a(1,1,0)/2, a_2 = a(1,0,1)/2,$ and $a_3 = a(0,1,1)/2$. The parity eigenvalues $\xi_\nu(\Gamma_i) (n = 1, \ldots, 4; i = 1, \ldots, 8)$ of the magnon eigenfunctions $|u_n(k)\rangle$ at the TRIM $\Gamma_i$ are numerically evaluated according to [47]

$$P_{\Gamma_i} |u_n(\Gamma_i)\rangle = \xi_\nu(\Gamma_i) |u_n(\Gamma_i)\rangle.$$  

(A5)

The relevant parity eigenvalues for “quarter” and “half filling” are given in Table I.

We now discuss selected examples of invariant surfaces $S_{abcd}$ and the respective products of parity eigenvalues (Fig. 5). Consider first $S_{1234}$ [Fig. 5(a)]; for $N_{12}$ (quarter filling), the parity eigenvalues at $\Gamma_1, \Gamma_2, \Gamma_3,$ and $\Gamma_4$ are given in the column “$\xi_1$” of Table I. All of these parity eigenvalues are positive, indicating that $S_{1234}$ is a trivial plane with respect to $N_{12}$: the nodal line does not pierce $S_{1234}$, which is in accordance with the graphical presentation in Fig. 5(a) (the blue line never touches the green surface). However, $N_{23}$ (red line) clearly pierces through $S_{1234}$ at $k$ and $-k$. Hence, we expect a negative parity eigenvalue product, which is corroborated by the numerics: An inspection of the column $\xi_1\xi_2$ in Table I tells that even parity is only present at $\Gamma_1$, while odd parity is found for the other TRIMs. This means that an invariant surface $S_{abcd}$ is nontrivial with respect to $N_{23}$ if and only if it contains $\Gamma_1$. This is the case for $S_{1234}$, $S_{1357}$, and $S_{1368}$ [Figs. 5(a), 5(b), and 5(c)], respectively, but not for $S_{3678}, S_{2356},$ and $S_{2378}$ [Figs. 5(d), 5(e), and 5(f)], respectively. Returning to $N_{12}$, we find odd parity only at $\Gamma_1$ (column $\xi_1$ in Table I). Thus, only the invariant surfaces containing $\Gamma_1$ exhibit a negative parity product [Figs. 5(b), 5(d), and 5(e)].

The $Z_2$ invariants $(v_0; v_1v_2v_3)$, defined by [28,48]

$$(-1)^{v_0} = \prod_{n_1=0,1} \xi_{n_1n_2n_3},$$  

(A6)

$$(-1)^{v_{(1,2,3)}} = \prod_{n_2=0,1} \xi_{n_1n_2n_3},$$  

(A7)

comprise the strong $(v_0)$ topological index and the weak topological indices $(v_{(1,2,3)})$ [48]. The $Z_2$ invariants of $N_{12}$ (quarter filling) read $(1; 111)$, those for $N_{23}$ (half filling) read $(1; 000)$. Thus, both nodal lines are strong $(v_0 = 1)$. The trivial weak indices for $N_{23}$ indicate that it is located around the $\Gamma$ point, while $N_{12}$ is located around an $L$ point.

We now address topological phase transitions, that is, the change of $(v_0; v_1v_2v_3)$ upon variation of $\lambda \in (0,1)$ [50]. It turns out that the phase $(1; 111)$ for quarter filling ($N_{12}$) is robust, while the phase $(1; 000)$ at half filling ($N_{23}$) changes into the weak phase $(0; 111)$ present for $\lambda < 0.618034$. This is understood from Fig. 6: for decreasing $\lambda$, the diameter of $N_{23}$...
increases until it touches the Brillouin zone boundary at the other $L$ points, labeled $L'$ (red and orange curves in Fig. 6).

Since the $L'$ points are TRIMs, the local parity eigenvalues of the second band at $\Gamma_{6,7,8}$ may change sign, and consequently, the $Z_2$ invariants may change as well. The further increase of $N_2$ causes a Lifshitz transition, i.e., it splits into two line nodes, both of which are open (yellow curve in Fig. 6) but nontrivial as we have checked numerically by the calculation of $\gamma_1[C(r)]$.

The above finding lends for a classification of nodal lines: open nodal lines are topologically weak ($\nu_0 = 0$) in the sense that the formation of an open or closed surface projection of a nodal line depends on the surface normal [compare Figs. 6(a) and 6(b)]. This is fully in line with electronic $Z_2$ topological insulators, for which $\nu_0 = 1$ guarantees a topological surface state on any surface. For $\nu_0 = 0$, the weak indices ($\nu_1 \nu_2 \nu_3 = (111)$) can then be understood as Miller indices, indicating a surface normal with nontrivial projection of the nodal line (closed loop); i.e., the [111] direction [cf. Fig. 6(a)].

[49] In a recent study, Su et al. considered anisotropy in combination with DMI and its effects on magnon Weyl points [38]. However, the authors did not discuss the case of zero DMI and did not identify nodal lines.

[50] \( \lambda > 1 \) \( (J'_N > J_N) \) allows for nodal lines as well. However, an analysis of this parameter range does not provide further insight.